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MINIMAX AND SEMI-MINIMAX ESTIMATORS FOR THE PARAMETER OF THE INVERTED EXPONENTIAL DISTRIBUTION UNDER GENERAL ENTROPY LOSS FUNCTION

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ABSTRACT
Datura stramonium was examined for its antimicrobial activity on some oral pathogens using methanol, chloroform, This paper is concerned with the problem of finding the minimax and semi-minimax estimators for the parameter of the inverted exponential distribution under general entropy loss function by applying the theorem of Lehmann. The performance of the obtained estimators along with the maximum likelihood estimator as a classical estimator have been compared empirically through simulation experiment with respect to their mean squared errors. Among the set of conclusions that have been reached, it is observed that, Bayes estimators under general entropy loss function introduce semi-minimax estimators corresponding to informative priors and introduce minimax estimators corresponding to non-informative priors. The performance of the minimax and semi-minimax estimators depends on the values of the hyper-parameters and constants that have had a significant impact on the performance of these estimators.

KEYWORDS: Inverted exponential distribution; maximum likelihood estimator; minimax estimator; semi-minimax estimator; Bayes estimator; informative prior; non-informative prior; general entropy loss function; mean squared errors.

INTRODUCTION

The inverted exponential distribution is a member of continuous probability distributions. It has been introduced by Keller and Kamath [6] in (1982). The probability density function and distribution function of inverted exponential distribution (from now onwards denoted as IED) are defined as [10]:

$$f(t;\theta) = \frac{1}{\theta t^2} e^{-1/\theta t} \; ; t > 0, \theta > 0 \qquad (1)$$

$$F(t;\theta) = e^{-1/\theta t} \quad ; \theta > 0 \qquad (2)$$

The IED has no finite moments where the r^{th} moment of the IED is given as [9]:

$$E(T^r) = \frac{1}{\theta^r}\Gamma(1-r) \; ; r < 1 \qquad (3)$$

Thus the expectation and the variance of the IED do not exist.

2. Maximum Likelihood Estimation

The maximum likelihood estimation method is one of the most widely used as classical estimation. Classical view is that there is some fixed (unknown) value of the parameter that is driving a process and, hence, its value is reflected in the data we see [4]. Assume that $(t_1, t_2, \dots, t_n)$ are the n independent random sample drawn from the IED defined by (1), then the likelihood function is obtained as:

$$L(\theta|\underline{t}) = \prod_{i=1}^{n} f(t_i,\theta) = \prod_{i=1}^{n} \frac{1}{\theta t^2} e^{-1/\theta t} \Rightarrow L(\theta|\underline{t}) = \frac{1}{\theta^n}\prod_{i=1}^{n} \frac{1}{t_i^2}\, e^{-S/\theta} \qquad (4)$$

Where: $S = \sum_{i=1}^{n} \frac{1}{t_i}$

The maximum likelihood estimator of θ, denoted by $\hat{\theta}_{ML}$, yields by taking the derivative of the natural log-likelihood function with respect to θ and setting it equal to zero:

$$\hat{\theta}_{ML} = \frac{S}{n} \quad ; S = \sum_{i=1}^{n} \frac{1}{t_i} \qquad (5)$$

3. Minimax Estimation

The minimax estimation is an upgraded non-classical approach in the estimation area of statistical inference, which was introduced by Wald (1950) from the concept of game theory. It opens a new dimension in statistical estimation and enriches the method of point estimations. According to Wald (1945), "mimimax approach tries to guard against the worst by requiring that the chosen decision rule should provide maximum protection against the highest possible risk". An estimator having this property is called a minimax estimator [11]. The derivation of minimax estimators depends basically on a theorem due to Hodge and Lehmann (1950) which can be stated as follows:

Lehmann's Theorem [8]**:** Let $\tau = \{F_\theta ; \theta \in \Theta\}$ be a family of distribution functions and D be a class of estimators of the parameterθ. Suppose that $d^* \in D$ is a Bayes estimator against a prior distribution $\pi(\theta)$ on the parameter space Θ. Then Bayes estimator d^* is said to be

minimax estimator if the risk function of d^* is independent on θ.
Mathematically, this theorem can be proved through applying two steps: first step is finding Bayes estimator $\hat{\theta}$ of θ while the second step is showing that the risk function of $\hat{\theta}$, $R(\hat{\theta},\theta)$, is a constant or not.

In order to obtain Bayes estimation of the parameter θ, we consider four priors "inverted gamma and gumbel type II as an informative prior distributions, Jeffrey and extension of Jeffrey as a non-informative prior distributions:

① **Inverted Gamma Prior:** The probability density function of inverted gamma prior is defined as [1]:

$$\pi_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)\theta^{\alpha+1}} e^{-\beta/\theta} \quad ; \theta > 0 \qquad (6)$$

Where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameter respectively of the prior distribution. The posterior density of (θ) corresponding to the prior $\pi_1(\theta)$ is obtained as:

$$H_1(\theta|\underline{t}) = \frac{\prod_{i=1}^n f(t_i,\theta)\pi_{1(\theta)}}{\int_\theta \prod_{i=1}^n f(t_i,\theta)\pi_{1(\theta)} d\theta} \Rightarrow H_1(\theta|\underline{t}) = \frac{(S+\beta)^{n+\alpha}}{\Gamma(n+\alpha)\theta^{n+\alpha+1}} e^{-(S+\beta)/\theta} \qquad (7)$$

Equation (7) implies that $(\theta|\underline{t}) \backsim$ Inverted Gamma $(n+\alpha\,, S+\beta)$.

② **Gumbel Type II Prior:** The probability density function of Gumbel type II prior defined as [1]:

$$\pi_2(\theta) = b\left(\frac{1}{\theta}\right)^2 e^{-b/\theta} \quad ; \theta > 0, b > 0 \qquad (8)$$

The posterior density of (θ) corresponding to the prior $\pi_2(\theta)$ is obtained as:

$$H_2(\theta|\underline{t}) = \frac{\prod_{i=1}^n f(t_i,\theta)\,\pi_2(\theta)}{\int_\theta \prod_{i=1}^n f(t_i,\theta)\,\pi_2(\theta) d\theta} \Rightarrow H_2(\theta|\underline{t}) = \frac{(S+b)^{n+1}}{\Gamma(n+1)\theta^{n+2}} e^{-(S+b)/\theta} \qquad (9)$$

Equation (9) implies that $(\theta|\underline{t}) \backsim$ Inverted Gamma $(n+1\,, S+b)$.

③_**Jeffrey's Prior:** Jeffrey's prior is proposed by Harold Jeffrey in (1946). It is based on Fisher information [7], such that: $\pi_3(\theta) \propto \sqrt{I(\theta)}$ Where $I(\theta) = -nE\left[\frac{\partial^2 \ln f(t,\theta)}{\partial^2\theta}\right]$ is the Fisher's information matrix. For the model (1),

$$\pi_3(\theta) = \frac{w\sqrt{n}}{\theta} \quad ; \theta > 0 \;, w{:}\, constant \qquad (10)$$

The posterior density of (θ) corresponding to the prior $\pi_3(\theta)$ is obtained as:

$$H_3(\theta|\underline{t}) = \frac{\prod_{i=1}^n f(t_i,\theta)\pi_3(\theta)}{\int_\theta \prod_{i=1}^n f(t_i,\theta)\pi_3(\theta) d\theta} \Rightarrow H_3(\theta|\underline{t}) = \frac{S^n}{\Gamma(n)\theta^{n+1}} e^{-S/\theta} \qquad (11)$$

Equation (11) implies that $(\theta|\underline{t}) \backsim$ Inverted Gamma $(n\,, S)$.

④_**Extension of Jeffrey's Prior:** The extension of Jeffrey's prior is considered as [2]:
$\pi_4(\theta) \propto [I(\theta)]^k \,; k \in R^+$ Where $I(\theta)$ is the Fisher's information matrix. For the model (1),

$$\pi_4(\theta) = \frac{wn^k}{\theta^{2k}} \quad ; \quad \theta > 0 \;, \; w{:}\, constant \qquad (12)$$

The posterior density of (θ) corresponding to the prior $\pi_4(\theta)$ is obtained as:

$$H_4(\theta|\underline{t}) = \frac{\prod_{i=1}^n f(t_i,\theta)\pi_4(\theta)}{\int_\theta \prod_{i=1}^n f(t_i,\theta)\pi_4(\theta) d\theta} \Rightarrow H_4(\theta|\underline{t}) = \frac{S^{n+2k-1}}{\Gamma(n+2k-1)\,\theta^{n+2k}} e^{-S/\theta} \qquad (13)$$

Equation (13) implies that $(\theta|\underline{t}) \backsim$ Inverted Gamma $(n+2k-1, S)$.

Let consider the general entropy loss function which was proposed by Calabria and Pulcini [3] as:

$$L(\hat{\theta}_G,\theta) = q\left[\left(\frac{\hat{\theta}_G}{\theta}\right)^c - c\ln\left(\frac{\hat{\theta}_G}{\theta}\right) - 1\right] \; ; \; q > 0, c \neq 0 \qquad (14)$$

Whose minimum occurs at $\hat{\theta}_G = \theta$ where $\hat{\theta}_G$ is an estimate of θ under general entropy loss function. Without loss of generality, we assume q=1. This loss is a generalization of the entropy loss used by several authors, where the value of the shape parameter c was taken as 1 [5]. When $c > 0$ then over estimation (positive error) causes more serious consequences than under estimation (negative error) and converse for $c < 0$. Bayes estimator of θ under general entropy loss function is obtained as:

$$\hat{\theta}_G = E_H(\theta^{-c}|\underline{t})]^{-1/c} \qquad (15)$$

Now, under general entropy loss function we obtain Bayesian estimators of the parameter θ corresponding to four posterior distributions as:

$$\hat{\theta}_{G_1} = \left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\right)^{1/c} (S+\beta) \qquad (16)$$

$$\hat{\theta}_{G_2} = \left(\frac{\Gamma(n+1)}{\Gamma(n+c+1)}\right)^{1/c} (S+b) \qquad (17)$$

$$\hat{\theta}_{G_3} = \left(\frac{\Gamma(n)}{\Gamma(n+c)}\right)^{1/c} S \qquad (18)$$

$$\hat{\theta}_{G_4} = \left(\frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)}\right)^{1/c} S \qquad (19)$$

The risk function $R(\hat{\theta},\theta)$ under general entropy loss function, (14) is:

$$R(\hat{\theta}_G,\theta) = E[L(\hat{\theta}_G,\theta)] = E\left[\left(\frac{\hat{\theta}_G}{\theta}\right)^c - c\ln\left(\frac{\hat{\theta}_G}{\theta}\right) - 1\right]$$

$$\Rightarrow R(\hat{\theta}_G,\theta) = \frac{1}{\theta^c}E(\hat{\theta}_G^c) - c\,E(\ln\hat{\theta}_G) + c\ln\theta - 1 \qquad (20)$$

▪ For $\hat{\theta}_{G_1}$ (16) the risk function (20) will be:

$$R(\hat{\theta}_{G_1},\theta) = \frac{1}{\theta^c}E\left(\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\right)^{1/c}(S+\beta)\right)^c - c\,E\left(\ln\left(\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\right)^{1/c}(S+\beta)\right)\right) + c\ln\theta - 1$$

$$\Rightarrow R(\hat{\theta}_{G_1},\theta) = \frac{1}{\theta^c}\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}E(S+\beta)^c - \ln\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)} - cE(\ln(S+\beta)) + c\ln\theta - 1 \qquad (21)$$

Now, we have to find $E(S+\beta)^c$ and $E(\ln(S+\beta))$.

$$E(S+\beta)^c = \int_0^\infty (S+\beta)^c f(s)\,dS = \int_0^\infty (S+\beta)^c\frac{S^{n-1}}{\Gamma(n)\theta^n}e^{-S/\theta}\,dS$$

Recall that : $(x+y)^n = \sum_{m=0}^{n}\binom{n}{m}x^m y^{n-m}$, therefore, $(S+\beta)^c = \sum_{n=0}^{c}\binom{c}{m}S^m\beta^{c-m}$

$$\Rightarrow E(S+\beta)^c = \sum_{m=0}^{c}\binom{c}{m}\frac{\beta^{c-m}}{\Gamma(n)\theta^n}\int_0^\infty S^{n+m-1}e^{-S/\theta}dS$$

$$\Rightarrow E(S+\beta)^c = \sum_{m=0}^{c}\binom{c}{m}\frac{\Gamma(n+m)}{\Gamma(n)}\beta^{c-m}\theta^m \qquad (22)$$

$$E(\ln(S+\beta)) = \int_0^\infty \ln(S+\beta)\,f(s)\,dS = \int_0^\infty \ln(S+\beta)\frac{S^{n-1}}{\Gamma(n)\theta^n}e^{-S/\theta}dS$$

Recall that : $\ln(1+x) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^m}{m}$, therefore, $\ln(S+\beta) = \ln(\beta) + \sum_{m=1}^{\infty}(-1)^n\frac{S^m}{m\,\beta^m}$

Then, $E(\ln(S+\beta)) = \ln(\beta) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m\,\Gamma(n)\beta^m\theta^n}\int_0^\infty S^{n+m-1}e^{-S/\theta}\,dS$

$$\Rightarrow E(\ln(S+\beta)) = \ln(\beta) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\Gamma(n+m)}{m\Gamma(n)}\left(\frac{\theta}{\beta}\right)^m \qquad (23)$$

Substituting (22) and (23) in (21) we get:

$$R(\hat{\theta}_{G_1},\theta) = \frac{1}{\theta^c}\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\sum_{m=0}^{c}\binom{c}{m}\frac{\Gamma(n+m)}{\Gamma(n)}\beta^{c-m}\theta^m - \ln\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)} - c\left(\ln\left(\frac{\beta}{\theta}\right) + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\Gamma(n+m)}{m\,\Gamma(n)}\left(\frac{\theta}{\beta}\right)^m\right) - 1 \qquad (24)$$

From (24), it's clear that $R(\hat{\theta}_{G_1},\theta)$ is not constant. So, $\hat{\theta}_{G_1}$ is not minimax estimator

Now, return to (21) and let $\beta \to o$, we get:

$$R(\hat{\theta}_{G_1},\theta) = \frac{\Gamma(n+\alpha)}{\theta^c\Gamma(n+\alpha+c)}E(S^c) - \ln\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\right) - c\,E(\ln(S)) + c\ln(\theta) - 1 \qquad (25)$$

Now, we have to find $E(S^c)$ and $E(\ln(S))$ …

$$E(S^c) = \int_0^\infty S^c\frac{S^{n-1}}{\Gamma(n)\theta^n}e^{-S/\theta}\,dS = \int_0^\infty \frac{S^{n+c-1}}{\Gamma(n)\theta^n}e^{-S/\theta}\,dS$$

$$\Rightarrow E(S^c) = \frac{\Gamma(n+c)}{\Gamma(n)}\theta^c \qquad (26)$$

$$E(\ln(S)) = \int_0^\infty \ln(S)\frac{S^{n-1}}{\Gamma(n)\theta^n}e^{-S/\theta}\,dS = \int_0^\infty \ln(S)\frac{S^{n-1}}{\Gamma(n)\theta^n}e^{-S/\theta}\,dS$$

$$\Rightarrow E(\ln(S)) = \ln(\theta) - \psi(n) \qquad (27)$$

Where: $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is a digamma function.

By Substituting (26) and (27) in (25) and simplification, we get:

$$R(\hat{\theta}_{G_1},\theta) = \frac{\Gamma(n+\alpha)\Gamma(n+c)}{\Gamma(n)\Gamma(n+\alpha+c)} - \ln\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+c)}\right) - c\,\psi(n) - 1 \qquad (28)$$

From (28), it's clear that $R(\hat{\theta}_{G_1},\theta)$ becomes constant. Therefore, $\hat{\theta}_{G_1}$ is semi-minimax estimator when $\beta \longrightarrow 0$.

▪ For $\hat{\theta}_{G_2}$ (17) the risk function (20) will be:

$$R(\hat{\theta}_{G_2},\theta) = \frac{1}{\theta^c}E\left(\left(\frac{\Gamma(n+1)}{\Gamma(n+c+1)}\right)^{1/c}(S+b)\right)^c - c\,E\left(\ln\left(\left(\frac{\Gamma(n+1)}{\Gamma(n+c+1)}\right)^{1/c}(S+b)\right)\right) + c\ln\theta - 1$$

$$\Rightarrow R(\hat{\theta}_{G_2},\theta) = \frac{1}{\theta^c}\frac{\Gamma(n+1)}{\Gamma(n+c+1)}E(S+b)^c - \ln\frac{\Gamma(n+1)}{\Gamma(n+c+1)} - c\,E(\ln(S+b)) + c\ln\theta - 1 \qquad (29)$$

Depending on replacement β by b in the equations (22) and (23), we get:

$$E(S+b)^c = \sum_{n=0}^{c} \binom{c}{m} \frac{\Gamma(n+m)}{\Gamma(n)} \; b^{c-m}\theta^m \qquad (30)$$

$$E(\ln(S+b)) = \ln(b) + \sum_{n=1}^{\infty} (-1)^{m+1} \frac{\Gamma(n+m)}{m\Gamma(n)} \left(\frac{\theta}{b}\right)^m \qquad (31)$$

By substituting (30) and (31) in (29) and simplification, we get:

$$R(\hat{\theta}_{G_2}, \theta) = \frac{1}{\theta^c} \frac{\Gamma(n+1)}{\Gamma(n+c+1)} \sum_{m=0}^{c} \binom{c}{m} \frac{\Gamma(n+m)}{\Gamma(n)} \; b^{c-m}\theta^m - \ln \frac{\Gamma(n+1)}{\Gamma(n+c+1)} -$$
$$c\left(\ln\left(\frac{b}{\theta}\right) + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\Gamma(n+m)}{m\,\Gamma(n)} \left(\frac{\theta}{b}\right)^m\right) - 1 \qquad (32)$$

From (32), it's clear that $R(\hat{\theta}_{G_2}, \theta)$ is not constant. Therefore, $\hat{\theta}_{G_2}$ is not minimax estimator.

Now, return to (29) and assume $b \longrightarrow o$ we get:

$$R(\hat{\theta}_{G_2}, \theta) = \frac{\Gamma(n+1)}{\theta^c \Gamma(n+c+1)} E(S^c) - \ln\left(\frac{\Gamma(n+1)}{\Gamma(n+c+1)}\right) - c\, E(\ln(S)) + c\ln(\theta) - 1 \qquad (33)$$

By substituting (26) and (27) in (33) and simplification, we get:

$$R(\hat{\theta}_{G_2}, \theta) = \frac{n}{n+c} - \ln\left(\frac{\Gamma(n+1)}{\Gamma(n+c+1)}\right) - c\,\psi(n) - 1 \qquad (34)$$

From (34), it's clear that $R(\hat{\theta}_{G_2}, \theta)$ becomes constant. Therefore, $\hat{\theta}_{G_2}$ is semi-minimax estimator when $b \longrightarrow 0$.

- For $\hat{\theta}_{G_3}$ (18) the risk function (20) will be:

$$R(\hat{\theta}_{G_3}, \theta) = \frac{1}{\theta^c} \frac{\Gamma(n)}{\Gamma(n+c)} E(S^c) - \ln \frac{\Gamma(n)}{\Gamma(n+c)} - c\, E(\ln(S)) + c\ln\theta - 1 \qquad (35)$$

Substituting (26) and (27) in (35), we get:

$$R(\hat{\theta}_{G_3}, \theta) = \frac{1}{\theta^c} \frac{\Gamma(n)}{\Gamma(n+c)} \frac{\Gamma(n+c)}{\Gamma(n)} \theta^c - \ln \frac{\Gamma(n)}{\Gamma(n+c)} - c\,\psi(n) - c\ln\theta + c\ln\theta - 1$$

$$R(\hat{\theta}_{G_3}, \theta) = \ln \frac{\Gamma(n+c)}{\Gamma(n)} - c\,\psi(n) \qquad (36)$$

From (36), it's clear that $R(\hat{\theta}_{G_3}, \theta)$ is constant. Therefore, $\hat{\theta}_{G_3}$ is minimax estimator.

- For $\hat{\theta}_{G_4}$ (19) the risk function (20) will be:

$$R(\hat{\theta}_{G_4}, \theta) = \frac{1}{\theta^c} E\left(\left(\left(\frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)}\right)^{1/c} S\right)^c\right) - c\, E\left(\ln\left(\left(\frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)}\right)^{1/c} S\right)\right) + c\ln\theta - 1$$

By simplification,

$$R(\hat{\theta}_{G_4}, \theta) = \frac{1}{\theta^c} \frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)} E(S^c) - \ln \frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)} - c\, E(\ln(S)) + c\ln\theta - 1 \qquad (37)$$

By substituting (26) and (27) in (37), we get:

$$R(\hat{\theta}_{G_4}, \theta) = \frac{\Gamma(n+c)\,\Gamma(n+2k-1)}{\Gamma(n)\,\Gamma(n+2k+c-1)} - \ln \frac{\Gamma(n+2k-1)}{\Gamma(n+2k+c-1)} - c\,\psi(n) - 1 \qquad (38)$$

From (38), it's clear that $R(\hat{\theta}_{G_4}, \theta)$ is constant. Therefore, $\hat{\theta}_{G_4}$ is minimax estimator.

4. SIMULATION EXPERIMENT AND RESULTS

To compare the estimators $\hat{\theta}_{G_1}, \hat{\theta}_{G_2}, \hat{\theta}_{G_3}, \hat{\theta}_{G_4}$ and $\hat{\theta}_{ML}$, we have considered the mean squared error (MSE) as statistical error criteria. The formulas that used to compute MSE is as follows:

$$MSE(\hat{\theta}) = \frac{\sum_{j=1}^{L} (\hat{\theta}_j - \theta)^2}{L} \qquad (39)$$

Where $\hat{\theta}_j$ is the estimate of θ at the j^{th} replicate (run).

The number of replication used was L = 3000 samples from the inverted exponential distribution of sizes n = 10, 15, 25, 30, 50 and 100 to represent small, medium, and large dataset. The values of the parameters chosen to be , θ = 1 and 3. The values of the hyper-parameters of inverted gamma and gumbel type II prior distributions chosen to be $(\alpha, \beta) = (4,4), (3,2), (6,10)$ and b=3,5. The constant of the extension of Jeffrey and general entropy loss function chosen to be k=1, 3 and c=± 2 , ± 5. The simulation program has been written by using MATLAB (R2011b) program. The results of Monte-Carlo simulation have been summarized in the tables (1)…(5).

TABLE 1: MSE Values for Maximum Likelihood Estimator of θ

θ	n=10	n=15	n=25	n=30	n=50	n=100
1	0.0972233	0.0628100	0.0396510	0.0331380	0.0202021	0.0100923
3	0.9071278	0.5845579	0.3802260	0.2878400	0.1792408	0.0924489

TABLE 2: MSE Values for Semi-Minimax Estimator of θ with Inverted Gamma Prior

n	α	β	[illegible]				[illegible]			
			c = -2	c = 2	c = -5	c = 5	c = -2	c = 2	c = -5	c = 5
10	4	4	0.0763055	0.0475996	0.1603476	0.0535516	0.6552433	0.8524260	0.7776950	1.0997933
	3	2	0.0753780	0.0661657	0.1428784	0.0830361	0.7304534	0.8856229	0.9835658	1.1325409
	6	10	0.1890804	0.0801322	0.3554071	0.0429012	0.4868861	0.6583398	0.5534524	0.8693329
15	4	4	0.0529962	0.0379923	0.0910773	0.0414255	0.4605355	0.5632788	0.5270229	0.7124686
	3	2	0.0526726	0.0483836	0.0813857	0.0582912	0.4990371	0.5755684	0.6192104	0.7202453
	6	10	0.1151840	0.0562515	0.1957822	0.0342602	0.3708881	0.4633734	0.4155175	0.5941372
25	4	4	0.0355812	0.0288227	0.0500289	0.0300781	0.3313124	0.3787240	0.3541101	0.4521738
	3	2	0.0355948	0.0334532	0.0461408	0.0377182	0.3480097	0.3833913	0.3880037	0.4527287
	6	10	0.0626941	0.0370154	0.0941185	0.0264093	0.2879067	0.3321627	0.3051831	0.3990795
30	4	4	0.0300926	0.0253548	0.0400398	0.0263661	0.2560790	0.2915187	0.2716506	0.3473823
	3	2	0.0302167	0.0288498	0.0373525	0.0321030	0.2668062	0.2933642	0.2939334	0.3457477
	6	10	0.0495690	0.0310667	0.0715826	0.0232423	0.2272470	0.2604493	0.2396492	0.3118128
50	4	4	0.0192193	0.0170656	0.0231002	0.0172822	0.1675432	0.1825434	0.1726759	0.2068864
	3	2	0.0191654	0.0183588	0.0219627	0.0195129	0.1717000	0.1830018	0.1807416	0.2054414
	6	10	0.0274529	0.0198155	0.0358862	0.0163151	0.1554717	0.1698492	0.1598792	0.1928716
100	4	4	0.0098387	0.0092590	0.0108207	0.0093141	0.0893320	0.0933683	0.0906968	0.1003136
	3	2	0.0098269	0.0096136	0.0105236	0.0099328	0.0904690	0.0934533	0.0927746	0.0997419
	6	10	0.0120767	0.0100177	0.0142302	0.0090218	0.0859453	0.0898945	0.0872081	0.0966333

TABLE 3: MSE Values for Semi-Minimax Estimator of θ with Gumbel Type II Prior

n	k	[illegible]				[illegible]			
		c = -2	c = 2	c = -5	c = 5	c = -2	c = 2	c = -5	c = 5
10	3	0.2426094	0.0901987	0.5756537	0.0582261	1.2446731	0.7018141	2.9073318	0.7370869
	5	0.4417408	0.1652946	0.9690057	0.0832517	1.4943034	0.6899900	3.5823841	0.6247367
15	3	0.1238456	0.0596253	0.2352414	0.0438777	0.7405935	0.4873542	1.3409950	0.5000543
	5	0.2085430	0.0955896	0.3793318	0.0562227	0.8532461	0.4860472	1.5953987	0.4448446
25	3	0.0612119	0.0383189	0.0943641	0.0316876	0.4285893	0.3420758	0.6057729	0.3497552
	5	0.0909222	0.0523141	0.1398673	0.0367373	0.4647115	0.3387549	0.6819308	0.3242326
30	3	0.0475705	0.0320344	0.0692482	0.0274324	0.3223714	0.2630309	0.4400305	0.2691880
	5	0.0678089	0.0417182	0.0995981	0.0308471	0.3482483	0.2615285	0.4923212	0.2515487
50	3	0.0258177	0.0199731	0.0331898	0.0179937	0.1904297	0.1700409	0.2287043	0.1730987
	5	0.0332807	0.0238509	0.0437709	0.0195433	0.1990835	0.1689854	0.2458292	0.1657591
100	3	0.0114769	0.0100335	0.0131904	0.0095148	0.0951963	0.0900312	0.1042877	0.0908106
	5	0.0133210	0.0110291	0.0157203	0.0099193	0.0973156	0.0897381	0.1083586	0.0888385

TABLE 4: MSE Values for Minimax Estimator of θ with Jeffrey's Prior

n	[illegible]				[illegible]			
	c = -2	c = 2	c = -5	c = 5	c = -2	c = 2	c = -5	c = 5
10	0.1655224	0.0908408	0.4133946	0.0982079	1.5564386	0.8420984	3.8593962	0.8652973
15	0.0891371	0.0601469	0.1638218	0.0632072	0.8546346	0.5523214	1.5797866	0.5606624
25	0.0487348	0.0385770	0.0700451	0.0396512	0.4658683	0.3694391	0.6625530	0.3761498
30	0.0391371	0.0324427	0.0528561	0.0333691	0.3454030	0.2805651	0.4734216	0.2862763
50	0.0225209	0.0198763	0.0272079	0.0200433	0.1983649	0.1768424	0.2385595	0.1798203
100	0.0106540	0.0100108	0.0117320	0.0100590	0.0971392	0.0918319	0.1064501	0.0926000

TABLE 5: MSE Values for Minimax Estimator of θ with Extension of Jeffrey's Prior

n	k	[illegible]				[illegible]			
		c = -2	c = 2	c = -5	c = 5	c = -2	c = 2	c = -5	c = 5
10	1	0.1105789	0.0911912	0.2276499	0.1105125	1.0368942	0.8331865	2.1367774	0.9954372
	3	0.1216260	0.1676789	0.0982079	0.2033237	1.0921071	1.4984229	0.8652973	1.8156654
15	1	0.0682280	0.0608259	0.1089308	0.0719448	0.6430432	0.5444617	1.0492142	0.6294533
	3	0.0782696	0.1071346	0.0632072	0.1314506	0.6818998	0.9294510	0.5606624	1.1423231
25	1	0.0416465	0.0386937	0.0545725	0.0432948	0.3994059	0.3684247	0.5199991	0.4072208
	3	0.0459861	0.0596996	0.0396512	0.0724284	0.4306022	0.5512724	0.3761498	0.6641522
30	1	0.0344543	0.0326297	0.0429051	0.0361097	0.3007974	0.2804053	0.3807760	0.3094464
	3	0.0381141	0.0485405	0.0333691	0.0584303	0.3267806	0.4183714	0.2862763	0.5060877
50	1	0.0207456	0.0198124	0.0238342	0.0210130	0.1835715	0.1772804	0.2095330	0.1894517
	3	0.0217537	0.0259664	0.0200433	0.0302748	0.1965590	0.2360265	0.1798203	0.2758150
100	1	0.0102259	0.0099962	0.0109577	0.0103223	0.0935326	0.0919269	0.0997437	0.0951829
	3	0.0105232	0.0117222	0.0100590	0.0130089	0.0970959	0.1082797	0.0926000	0.1201326

5. CONCLUSIONS & RECOMMENDATIONS

The most important conclusions and recommendations based on simulation experiment are:

1. Depending on MSE values, the maximum likelihood method gives estimate values better than Bayes method for all sample sizes corresponding to:
   - Inverted gamma prior with $(\alpha = 6, \beta = 10)$ for negative values of (c) when $\theta = 1$ as well as with $\alpha = \beta = 4$ for$(c = 5)$ and with $(\alpha = 3, \beta = 2)$ for $(c = \pm 5)$ when $(\theta = 3)$.
   - Gumbel type II prior with different values of (b) and negative values of (c) when $(\theta = 1,3)$ while with $(b = 5, c = 2)$ when$(\theta = 1)$.
   - Jeffrey's prior based on general entropy (with negative values) for different values of (θ).
   - Extension of Jeffrey's prior with $(k = 1)$ for negative and large positive values of (c) as well as with $(k = 3)$ except with $(c = -5)$.
2. Semi-minimax estimate for the parameter of IED corresponding to inverted gamma prior with hyper-parameters $(\alpha = 6 < \beta = 10)$ under general entropy loss function with large positive or large negative values of (c) represent the best estimate when $(\theta = 1)$ and $(\theta = 3)$ respectively for different sample sizes.
3. Informative Gumbel type II prior doesn't record any appearance as best prior with $\theta = 1$. While with $\theta = 3$, record appearance when $b = 5$ with positive values of c. It is important to mention that for small positive value of c, $c = 2$, Gumbel type II prior when $b = 5$ is the best prior for large sample sizes.
4. Between non-informative prior distributions, Jeffrey's prior doesn't record any appearance as best prior while extension of Jeffrey's prior with $(k = 3)$ record appearance as best prior for one time under general entropy loss function with $(c = -5)$ and that appearance disappear with the large value of θ , $\theta = 3$.
5. With two values of the parameter θ, the MSE values associated with semi-minimax estimates corresponding to Gumbel type II prior under general entropy loss function (with negative values of c) are increase as hyper-parameter value, b, increase.
6. The MSE values associated with all estimates corresponding to different values of θ and (c) except with$(c = -5)$, are increases as extension constant (k) increases.
7. The MSE values associated with all estimates reduce with the increase in the sample size and this conforms to the statistical theory. For large sample size$(n = 100)$, all the estimators have approximately the same MSE values.
8. In some sense, Jeffrey's prior and extension of Jeffrey's prior for specific values of (k) can get equivalent estimates based on general entropy loss function, such that, Bayes estimates corresponding to Jeffrey's prior under general entropy loss function with $(c = 5)$ are equivalent to that estimates with $(c = -5)$ corresponding to extension of Jeffrey's prior with $(k = 3)$
9. Minimax estimates corresponding to Jeffrey's prior, with the assumption of general entropy loss function with $(c = 2)$ are having the minimum MSE values for all sample sizes and values of θ.
10. Bayes estimators under general entropy loss function introduce semi-minimax estimators corresponding to informative priors and introduce minimax estimators corresponding to non-informative priors. The performance of the minimax and semi-minimax estimators depends on the values of the hyper-parameters and constants that have had a significant impact on the performance of these estimators.
11. With two values of θ, the performance of Bayes estimates under general entropy loss function with positive values of (c) is better comparing to negative values with all sample sizes corresponding to Gumbel type II, Jeffrey and extension of Jeffrey (with $k = 1$) priors.

In the light of the conclusions that being obtained for the estimation of the parameter of IED, some recommendations have been put forward:

1. Using inverted gamma prior with hyper-parameters $(\alpha = 6 < \beta = 10)$ as an appropriate prior distribution under general entropy loss function with large positive value of (c) for the situation that$(\theta = 1)$.
2. Using Gumbel type II with $(b = 5)$ as an appropriate prior distribution under general entropy with large positive value of (c) for the situation that$(\theta = 3)$.
3. Using extension of Jeffrey with $(k = 3)$ as an appropriate prior distribution under general entropy with small negative value of (c) for the situation that$(\theta = 1)$.

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