



ON THE MAXIMUM LIKELIHOOD, BAYES AND EMPIRICAL BAYES ESTIMATION FOR THE SHAPE PARAMETER, RELIABILITY AND FAILURE RATE FUNCTIONS OF KUMARASWAMY DISTRIBUTION

Nadia H. Al-Noor & Sudad K. Ibraheem

College of Science, Department of Mathematics / AL- Mustansiriyah University /Baghdad / Iraq.

ABSTRACT

This paper is considered with Kumaraswamy distribution. The maximum likelihood, Bayes and empirical Bayes methods of estimation are used for estimating the unknown shape parameter, reliability and failure rate functions of Kumaraswamy distribution under complete samples assuming that the other shape parameter is known. The performance of estimators is showed by demonstrating some numerical illustrations through Monte Carlo simulation study.

KEYWORDS: Kumaraswamy distribution; Maximum likelihood estimation; Bayes estimation; Empirical Bayes estimation; Squared error loss function; Precautionary loss function; Inverse Levy distribution.

INTRODUCTION

A two-parameter Kumaraswamy distribution on $(0, 1)$ has been proposed by Kumaraswamy (1980)^[10]. The

probability density and cumulative distribution functions of a Kumaraswamy distributed random variable are given by^[7]:

$$f(t; \theta, \lambda) = \theta \lambda t^{\lambda-1} (1 - t^\lambda)^{\theta-1} \quad ; 0 < t < 1 \quad (1)$$

$$F(t; \theta, \lambda) = 1 - (1 - t^\lambda)^\theta \quad ; 0 < t < 1 \quad (2)$$

Where $\theta > 0$ and $\lambda > 0$ are the shape parameters. The corresponding reliability function, $R(t)$, and failure rate function, $h(t)$, at mission time t are given as [7]:

$$R(t) = 1 - F(t; \theta, \lambda) = (1 - t^\lambda)^\theta \quad ; 0 < t < 1; \theta, \lambda > 0 \quad (3)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\theta \lambda t^{\lambda-1}}{1 - t^\lambda} \quad ; 0 < t < 1; \theta, \lambda > 0 \quad (4)$$

Kumaraswamy distribution (KD) is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures, hydrological data such as daily rain fall, daily stream flow, etc.^[12] Ponnambalam *et al.* (2001)^[11] pointed out that KD can be used to approximate many distributions, such as uniform, triangular, or almost any single model distribution and can also reproduce results of beta distribution depending on the choice of the shape parameters. Jones (2009)^[9] discussed the basic properties of KD. Garg (2009)^[6] considered the generalized order statistics from KD. Cordeiro *et al.* (2010)^[4] introduced and studied some mathematical properties of the Kumaraswamy Weibull distribution as a quite flexible model in analyzing positive data. Gholizadeh *et al.* (2011)^[7] studied the maximum likelihood and Bayes estimators for the shape parameter, reliability and failure rate functions of the KD in the cases of progressively type II censored samples. Gholizadeh *et al.* (2011)^[8] obtained

classical and Bayesian estimators for the shape parameter of the Kumaraswamy distribution using grouped and ungrouped data under three different loss functions. Cordeiro *et al.* (2012)^[3] proposed a new distribution referred to as Kumaraswamy Gumbel distribution and provide a comprehensive treatment of its structural properties. Sindhu *et al.* (2013)^[12] obtained the maximum likelihood and Bayes estimators using asymmetric loss functions for the shape parameter of KD under type-II censored samples. Eldin *et al.* (2014)^[5] derived the maximum likelihood and Bayes estimators using squared error loss function for the two shape parameters of KD based on general progressive type II censoring.

Maximum Likelihood Estimator (MLE): Let $\underline{t} = (t_1, \dots, t_n)$ be the life time of a random sample of size n drawn independently from KD defined by (1). The likelihood functions for the given sample observations are defined as:

$$L(\theta, \lambda | \underline{t}) = \prod_{i=1}^n f(t_i | \theta, \lambda) = \theta^n \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i)} e^{(\theta-1) \sum_{i=1}^n \ln(1-t_i^\lambda)} \quad (5)$$

The MLE of the unknown shape parameter, θ , denoted by $\hat{\theta}_{ML}$, assuming that the other shape parameter, λ , is known

yields by taking the derivative of the natural log-likelihood function with respect to θ and setting it equal to zero as:

$$\hat{\theta}_{ML} = \frac{n}{S} ; S = - \sum_{i=1}^n \ln(1 - t_i^\lambda) \quad (6)$$

The MLEs of $R(t)$ and $h(t)$, based on the invariant property of the MLE, are defined as:

$$R(t) = (1 - t^\lambda)^{\hat{\theta}_{ML}} \quad (7)$$

$$h(t) = \frac{\hat{\theta}_{ML} \lambda t^{\lambda-1}}{1 - t^\lambda} \quad (8)$$

Where $\hat{\theta}_{ML}$ as in (6).

Bayes Estimator (BE): Bayes approach is concerned with generating the posterior distribution of the unknown parameter given both the data and some prior density for this parameter. The posterior density function of unknown

parameter, θ , results by combining the likelihood function, $L(\theta, \lambda | \underline{t})$, with the density function of prior distribution, $g(\theta)$, as:

$$\pi(\theta | \underline{t}) = \frac{L(\theta, \lambda | \underline{t}) g(\theta)}{\int_{\theta} L(\theta, \lambda | \underline{t}) g(\theta) d\theta} \quad (9)$$

Bayes estimation, depending on the posterior distribution, is based in minimization of a Bayesian loss (risk) function which defined as an average cost-of-error function. There are two types of loss function: symmetric and asymmetric loss functions. The symmetric loss function associates equal importance to the losses due to overestimation and

underestimation of equal magnitude [2]. We have been adopted the squared error (SELF) loss function and precautionary loss function (PLF) to represent the two types of loss functions respectively. The SELF and PLF can be expressed respectively as [1]:

$$L_S(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad (10)$$

$$L_P(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}} \quad (11)$$

So, Bayes estimators of θ , $R(t)$ and $h(t)$ based on SELF, denoted by $\hat{\theta}_{BS}$, $\hat{R}(t)_{BS}$ and $\hat{h}(t)_{BS}$ can be obtained as:

$$\hat{\theta}_{BS} = E_{\pi}(\theta | \underline{t}) \Rightarrow \hat{\theta}_{BS} = \int_{\theta} \theta \pi(\theta | \underline{t}) d\theta \quad (12)$$

$$\hat{R}(t)_{BS} = E_{\pi}(R(t) | \underline{t}) \Rightarrow \hat{R}(t)_{BS} = \int_{\theta} R(t) \pi(\theta | \underline{t}) d\theta \quad (13)$$

$$\hat{h}(t)_{BS} = E_{\pi}(h(t) | \underline{t}) \Rightarrow \hat{h}(t)_{BS} = \int_{\theta} h(t) \pi(\theta | \underline{t}) d\theta \quad (14)$$

And Bayes estimator of θ , $R(t)$ and $h(t)$ based on PLF, denoted by $\hat{\theta}_{BP}$, $\hat{R}(t)_{BP}$ and $\hat{h}(t)_{BP}$, can be obtained as:

$$\hat{\theta}_{BP} = \sqrt{E_{\pi}(\theta^2 | \underline{t})} \Rightarrow \hat{\theta}_{BP} = \left[\int_{\theta} \theta^2 \pi(\theta | \underline{t}) d\theta \right]^{\frac{1}{2}} \quad (15)$$

$$\hat{R}(t)_{BP} = \sqrt{E_{\pi}(R^2(t) | \underline{t})} \Rightarrow \hat{R}(t)_{BP} = \left[\int_{\theta} R^2(t) \pi(\theta | \underline{t}) d\theta \right]^{\frac{1}{2}} \quad (16)$$

$$\hat{h}(t)_{BP} = \sqrt{E_{\pi}(h^2(t) | \underline{t})} \Rightarrow \hat{h}(t)_{BP} = \left[\int_{\theta} h^2(t) \pi(\theta | \underline{t}) d\theta \right]^{\frac{1}{2}} \quad (17)$$

Now, under the assumption of inverse Levy informative prior of θ with hyper parameter 'b' which is defined as [2]:

$$g(\theta) = \sqrt{\frac{b}{2\pi}} \theta^{-\frac{1}{2}} e^{-\frac{b\theta}{2}} ; \theta > 0 ; b > 0 \quad (18)$$

The posterior distribution of the unknown parameter θ of KD have been obtained by combining (9) with (18) as:

$$\pi(\theta|\underline{t}) = \frac{\theta^n \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i)} e^{(\theta-1) \sum_{i=1}^n \ln(1-t_i^\lambda)} \sqrt{\frac{b}{2\pi}} \theta^{-\frac{1}{2}} e^{-\frac{b\theta}{2}}}{\int_0^\infty \theta^n \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i)} e^{(\theta-1) \sum_{i=1}^n \ln(1-t_i^\lambda)} \sqrt{\frac{b}{2\pi}} \theta^{-\frac{1}{2}} e^{-\frac{b\theta}{2}} d\theta}$$

$$\Rightarrow \pi(\theta|\underline{t}) = \frac{\theta^{n-\frac{1}{2}} e^{-\theta(\frac{b}{2}+S)}}{\int_0^\infty \theta^{n-\frac{1}{2}} e^{-\theta(\frac{b}{2}+S)} d\theta}$$

By using the transformation, $y = \theta \left(\frac{b}{2} + S\right) \Rightarrow \theta = \frac{y}{\frac{b}{2}+S} \Rightarrow d\theta = \frac{dy}{\frac{b}{2}+S}$, we can get the final formula of $\pi(\theta|\underline{t})$ as:

$$\pi(\theta|\underline{t}) = \frac{\left(\frac{b}{2} + S\right)^{n+\frac{1}{2}}}{\Gamma\left(n + \frac{1}{2}\right)} \theta^{n-\frac{1}{2}} e^{-\theta(\frac{b}{2}+S)}; S = -\sum_{i=1}^n \ln(1-t_i^\lambda) \quad (19)$$

from (19), we conclude that: $(\theta|\underline{t}) \sim \text{Gamma}\left(n + \frac{1}{2}, \frac{b}{2} + S\right)$.

Now, by using the expression in (12), (13) and (14), Bayes estimators of θ , $R(t)$ and $h(t)$ based on SELF will be:

$$\hat{\theta}_{BS} = \frac{n + \frac{1}{2}}{\frac{b}{2} + S} \quad (20)$$

$$\hat{R}(t)_{BS} = \left(1 - \frac{\ln(1-t^\lambda)}{\frac{b}{2} + S}\right)^{-(n+\frac{1}{2})} \quad (21)$$

$$\hat{h}(t)_{BS} = \frac{\left(n + \frac{1}{2}\right) (\lambda t^{\lambda-1})}{\left(\frac{b}{2} + S\right) (1-t^\lambda)} \quad (22)$$

As well as, by using the expression in (15), (16) and (17), Bayes estimators of θ , $R(t)$ and $h(t)$ based on PLF will be:

$$\hat{\theta}_{BP} = \frac{\sqrt{\left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right)}}{\frac{b}{2} + S} \quad (23)$$

$$\hat{R}(t)_{BP} = \left(1 - \frac{2\ln(1-t^\lambda)}{\frac{b}{2} + S}\right)^{-\frac{1}{2}(n+\frac{1}{2})} \quad (24)$$

$$\hat{h}(t)_{BP} = \frac{\sqrt{\left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right)} (\lambda t^{\lambda-1})}{\left(\frac{b}{2} + S\right) (1-t^\lambda)} \quad (25)$$

Empirical Bayes Estimators (BE): Bayes estimators of θ , $R(t)$ and $h(t)$ are seen to depend upon the hyper-parameter 'b'. If 'b' is unknown, then we may use the empirical Bayes approach to get its estimate from

likelihood function and probability density function of prior distribution. The marginal probability density function of T can calculate from (4) and (9) as:

$$f(\underline{t}|\theta) = \int_0^\infty \theta^n \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i)} e^{(\theta-1) \sum_{i=1}^n \ln(1-t_i^\lambda)} \sqrt{\frac{b}{2\pi}} \theta^{-\frac{1}{2}} e^{-\frac{b\theta}{2}} d\theta$$

$$f(\underline{t}|\theta) = \sqrt{\frac{b}{2\pi}} \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i) + S} \int_0^\infty \theta^{n-\frac{1}{2}} e^{-\theta(\frac{b}{2}+S)} d\theta$$

By using the transformation, $y = \theta \left(\frac{b}{2} + S\right) \Rightarrow \theta = \frac{y}{\frac{b}{2}+S} \Rightarrow d\theta = \frac{dy}{\frac{b}{2}+S}$, we can get the final formula of $f(\underline{t}|\theta)$ as:

$$\pi(\theta|\underline{t}) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\left(\frac{b}{2} + S\right)^{n+\frac{1}{2}}} \sqrt{\frac{b}{2\pi}} \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln(t_i) + S}; S = -\sum_{i=1}^n \ln(1-t_i^\lambda) \quad (26)$$

The MLE of b based on $f(\underline{t}|b)$, denoted by \hat{b}_{ML} which is calculating by taking the derivative of the natural log for (26) and setting it equal to zero, is:

$$\hat{b}_{ML} = \frac{S}{n} ; S = - \sum_{i=1}^n \ln(1 - t_i^\lambda) \quad (27)$$

The empirical Bayes estimators of θ , $R(t)$ and $h(t)$ based on SELF and PLF, denoted by $\hat{\theta}_{EBS}$, $\hat{R}(t)_{EBS}$, $\hat{h}(t)_{EBS}$, $\hat{\theta}_{EBP}$, $\hat{R}(t)_{EBP}$ and $\hat{h}(t)_{EBP}$ respectively can be obtained

by replacing the hyper-parameter b appears in (20), (21), (22), (23), (24) and (25) by \hat{b}_{ML} as:

$$\hat{\theta}_{EBS} = \frac{n + \frac{1}{2}}{\frac{\hat{b}_{ML}}{2} + S} \quad (28)$$

$$\hat{R}(t)_{EBS} = \left(1 - \frac{\ln(1 - t^\lambda)}{\frac{\hat{b}_{ML}}{2} + S} \right)^{-(n + \frac{1}{2})} \quad (29)$$

$$\hat{h}(t)_{EBS} = \frac{(n + \frac{1}{2}) (\lambda t^{\lambda-1})}{\left(\frac{\hat{b}_{ML}}{2} + S \right) (1 - t^\lambda)} \quad (30)$$

$$\hat{\theta}_{EBP} = \frac{\sqrt{(n + \frac{1}{2})(n + \frac{3}{2})}}{\frac{\hat{b}_{ML}}{2} + S} \quad (31)$$

$$\hat{R}(t)_{EBP} = \left(1 - \frac{2 \ln(1 - t^\lambda)}{\frac{\hat{b}_{ML}}{2} + S} \right)^{-\frac{1}{2}(n + \frac{1}{2})} \quad (32)$$

$$\hat{h}(t)_{EBP} = \frac{\sqrt{(n + \frac{1}{2})(n + \frac{3}{2})} (\lambda t^{\lambda-1})}{\left(\frac{\hat{b}_{ML}}{2} + S \right) (1 - t^\lambda)} \quad (33)$$

Simulation Study and Results

Monte Carlo simulation study has been conducted to compare the performance of the estimators that we obtained in the previous sections for the shape parameter of KD with respect to their mean square error (MSE) as well as reliability and failure rate functions of KD with respect to their integrated mean square error (IMSE). A

random samples of different size $n = 10, 30$ and 50 were generated independently from KD through the adoption of inverse transformation method based on different default values of the parameter θ , $\theta = 1, 2, 3$ and λ fixed at 2 . where:

$$MSE(\hat{\theta}) = \frac{\sum_{j=1}^L (\hat{\theta}_j - \theta)^2}{L} \quad (34)$$

$$IMSE(\hat{R}(t)) = \frac{1}{L} \sum_{j=1}^L \left(\frac{1}{n_t} \sum_{i=1}^{n_t} (\hat{R}_j(t_i) - R(t_i))^2 \right) \quad (35)$$

$$IMSE(\hat{h}(t)) = \frac{1}{L} \sum_{j=1}^L \left(\frac{1}{n_t} \sum_{i=1}^{n_t} (\hat{h}_j(t_i) - h(t_i))^2 \right) \quad (36)$$

$\hat{\theta}_j$: is the estimate of θ at the j^{th} replicate (run).

L : is the number of sample replicated chosen to be (3000).

n_t : is the number of times chosen to be (4) where $(t = 0.2, 0.4, 0.6, 0.8)$.

$\hat{R}_j(t_i)$: is the estimates of $R(t)$ at the j^{th} replicate (run) and i^{th} time.

$\hat{h}_j(t_i)$: is the estimates of $h(t)$ at the j^{th} replicate (run) and i^{th} time.

The simulation program has been written by using MATLAB (R2011b) program. Tables (1)-(3) summarized the results of Monte-Carlo simulation.

TABLE 1: Values of Statistical Criteria for Maximum Likelihood, Bayes and Empirical Bayes Estimates with $\theta = 1, \lambda = 2$

		1, 2			
ical	Meth.	Est.	MSE	IMSE	IMSE
10	ML	1.1060858	0.1630172	0.0064901	0.2000685
	BS	1.1265234	0.1616434	0.0057561	0.1983824
		1.0939354	0.1365334	0.0053185	0.1675653
		1.0633823	0.1167302	0.0049869	0.1432611
	BP	1.1789476	0.1915275	0.0053576	0.2350586
		1.1448430	0.1608518	0.0050403	0.1974109
		1.1128681	0.1361866	0.0048204	0.1671397
	EBS	1.1060858	0.1630172	0.0059424	0.2000685
	EBP	1.1575589	0.1910415	0.0056087	0.2344622
	Best Estimate				
30	ML	1.0255431	0.0384335	0.0023092	0.0471688
	BS	1.0334885	0.0388362	0.0022337	0.0476630
		1.0245058	0.0370386	0.0021885	0.0454569
		1.0156829	0.0354638	0.0021526	0.0435242
	BP	1.0502943	0.0414807	0.0021876	0.0509087
		1.0411656	0.0393274	0.0021552	0.0482659
		1.0321992	0.0374094	0.0021317	0.0459119
	EBS	1.0255431	0.0384335	0.0022509	0.0471688
	EBP	1.0422197	0.0408023	0.0022154	0.0500760
	Best Estimate				
50	ML	1.0165207	0.0274390	0.0019666	0.0336754
	BS	1.0213589	0.0276446	0.0019359	0.0339277
		1.0160880	0.0269368	0.0019158	0.0330591
		1.0108723	0.0262987	0.0018989	0.0322760
	BP	1.0314218	0.0287141	0.0019137	0.0352403
		1.0260989	0.0278874	0.0018982	0.0342258
		1.0208319	0.0271329	0.0018859	0.0332998
	EBS	1.0165207	0.0274390	0.0019402	0.0336754
	EBP	1.0265359	0.0284081	0.0019221	0.0348648
	Best Estimate				

TABLE 2: Values of Statistical Criteria for Maximum Likelihood, Bayes and Empirical Bayes Estimates with $\theta = 2, \lambda = 2$

		2, 2			
ical	Meth.	Est.	MSE	IMSE	IMSE
10	ML	2.1918298	0.5993581	0.0064661	0.7355829
	BS	2.1696043	0.5049642	0.0055958	0.6197347
		2.0534796	0.3771255	0.0053344	0.4628402
		1.9501600	0.3021721	0.0054186	0.3708511
	BP	2.2705695	0.5947587	0.0057632	0.7299381
		2.1490408	0.4321229	0.0057917	0.5303377
		2.0409131	0.3299037	0.0061400	0.4048856
	EBS	2.1918298	0.5993581	0.0062113	0.7355829
	EBP	2.2938293	0.7024722	0.0063334	0.8621332
	Best Estimate				
30	ML	2.0695718	0.1524822	0.0022222	0.1871390
	BS	2.0672066	0.1464201	0.0021098	0.1796991
		2.0316591	0.1331499	0.0020641	0.1634128
		1.9973513	0.1232430	0.0020570	0.1512542
	BP	2.1008219	0.1567210	0.0021229	0.1923413
		2.0646963	0.1406659	0.0021141	0.1726371
		2.0298307	0.1281664	0.0021429	0.1572966
	EBS	2.0695718	0.1524822	0.0021825	0.1871390
	EBP	2.1032256	0.1631382	0.0021936	0.2002169
	Best Estimate				
50	ML	2.0465293	0.0885210	0.0013487	0.1086404
	BS	2.0456354	0.0865154	0.0013041	0.1061790
		2.0247219	0.0815762	0.0012837	0.1001172
		2.0042401	0.0776943	0.0012771	0.0953530

BP	$\theta = 0.5$	2.0657899	0.0904331	0.0013059	0.1109871
	$b = 1.5$	2.0446704	0.0845637	0.0012991	0.1037838
EBS	$\theta = 0.5$	2.0239868	0.0797899	0.0013060	0.0979249
	$b = 1.5$	2.0465293	0.0885210	0.0013309	0.1086404
EBP		2.0666927	0.0925140	0.0013322	0.1135409
Best Estimate			$B_{0.925}^{0.888}$ $b = 1.5$	$B_{0.013}^{0.013}$ $b = 1.5$	$B_{1.135}^{1.086}$ $b = 1.5$

TABLE 3: Values of Statistical Criteria for Maximum Likelihood, Bayes and Empirical Bayes Estimates with $\theta = 3, \lambda = 2$

Meth.		Est.	MSE	IMSE	IMSE	
10	ML	3.3479368	1.5200942	0.0064312	1.8655878	
	BS	$\theta = 0.5$	3.2160542	1.0854891	0.0054156	1.3322038
		$b = 1.5$	2.9679986	0.7358998	0.0053782	0.9031583
	BP	$\theta = 0.5$	2.7581148	0.5974546	0.0059939	0.7332467
		$b = 1.5$	3.3657173	1.2714931	0.0058441	1.5604835
	EBS	$\theta = 0.5$	3.1061181	0.8161250	0.0062434	1.0016174
		$b = 1.5$	2.8864671	0.6031641	0.0072510	0.7402539
	EBP	3.3479368	1.5200942	0.0062228	1.8655878	
	Best Estimate	3.5037372	1.7860266	0.0064528	2.1919625	
			$B_{0.520}^{0.786}$ $b = 1.5$	$B_{0.062}^{0.064}$ $b = 1.5$	$B_{1.865}^{0.655}$ $b = 1.5$	
30	ML	3.0924569	0.3540775	0.0021035	0.4345538	
	BS	$\theta = 0.5$	3.0623097	0.3238884	0.0020043	0.3975031
		$b = 1.5$	2.9848991	0.2879197	0.0020042	0.3533593
	BP	$\theta = 0.5$	2.9114277	0.2672952	0.0020807	0.3280472
		$b = 1.5$	3.1121067	0.3430658	0.0020693	0.4210392
	EBS	$\theta = 0.5$	3.0334372	0.2982422	0.0021239	0.3660280
		$b = 1.5$	2.9587710	0.2696565	0.0022533	0.3309452
	EBP	3.0924569	0.3540775	0.0020918	0.4345538	
	Best Estimate	3.1427441	0.3772339	0.0021366	0.4629733	
			$B_{0.354}^{0.377}$ $b = 1.5$	$B_{0.020}^{0.021}$ $b = 1.5$	$B_{0.434}^{0.462}$ $b = 1.5$	
50	ML	3.0513756	0.1890061	0.0012037	0.2319642	
	BS	$\theta = 0.5$	3.0346778	0.1796576	0.0011709	0.2204909
		$b = 1.5$	2.9889185	0.1678137	0.0011726	0.2059552
	BP	$\theta = 0.5$	2.9445440	0.1608136	0.0012022	0.1973640
		$b = 1.5$	3.0645769	0.1861589	0.0011959	0.2284700
	EBS	$\theta = 0.5$	3.0183667	0.1713489	0.0012179	0.2102938
		$b = 1.5$	2.9735550	0.1615611	0.0012672	0.1982814
	EBP	3.0513756	0.1890061	0.0012006	0.2319642	
	Best Estimate	3.0814392	0.1966894	0.0012185	0.2413938	
			$B_{0.189}^{0.196}$ $b = 1.5$	$B_{0.012}^{0.012}$ $b = 1.5$	$B_{0.241}^{0.241}$ $b = 1.5$	

CONCLUSION & RECOMMENDATION

From table (1) with $\theta = 1$ and $\lambda = 2$, it appears that:

- The maximum likelihood, Bayes and empirical Bayes estimation methods gives estimate values greater than the default value for the shape parameter θ of KD (i.e. overestimate values) with all sample sizes under study.
- Bayes estimation method with hyper-parameter $b = 1.5$ based on square error loss function, BS, recorded the best estimate of the shape parameter and failure rate function of KD for all sample sizes under study.
- Bayes estimation method with hyper-parameter $b = 1.5$ based on precautionary loss function, BP, recorded the best estimate of the reliability function of KD for all sample sizes under study.
- Increase the value of hyper-parameter b , decreasing the values of MSE and IMSE associated with BS and BP.
- With Bayes and empirical Bayes methods, using square error loss function to estimate the shape parameter and failure rate function of KD is better than using

precautionary loss function while the reverse is true to estimate the reliability function.

From table (2) with $\theta = 2$ and $\lambda = 2$, it appears that:

- The maximum likelihood, Bayes and empirical Bayes methods gives overestimate values with all sample sizes under study except BS with $b = 1.5$ for small and moderate sample sizes $n = 10, 30$ which were underestimate values.
- BS with $b = 1.5$, recorded the best estimate of the shape parameter, failure rate and reliability functions for all sample sizes except $n = 10$ when BS with $b = 1$, recorded the best estimate of the reliability function.
- Increase the value of b , decreasing the values of MSE and IMSE associated with BS and BP to estimate the shape parameter and failure rate function.
- With Bayes and empirical Bayes methods, using square error loss function is better than using precautionary loss function to estimate the shape parameter, failure rate and reliability functions of KD.

From table (3) with $\theta = 3$ and $\lambda = 2$, it appears that:

- The maximum likelihood, Bayes and empirical Bayes methods gives overestimate values with all sample sizes under study except BS with $b = 1, 1.5$ and BP with $b = 1.5$ for all sample sizes which were underestimate values.
- BS with $b = 1.5$, recorded the best estimate of the shape parameter and failure rate function for all sample sizes as well as BS recorded the best estimate of the reliability function with $b = 1$ for $n = 10, 30$ and with $b = 0.5$ for $n = 50$.
- Increase the value of b , decreasing the values of MSE and IMSE associated with BS and BP to estimate the shape parameter and failure rate function.
- Increase the value of b , increasing the values of IMSE associated with BS and BP to estimate the reliability function for $n = 50$.
- With Bayes and empirical Bayes methods, using square error loss function is better than using precautionary loss function to estimate the shape parameter, failure rate and reliability functions of KD.

From tables (1), (2) and (3), it appears that:

- The performance of ML and EBS to estimate the shape parameter and failure rate function of KD are identical while the performance of EBS is better than ML to estimate the reliability function with all sample sizes under study.
- The MSE and IMSE values are decreasing as the sample sizes increasing.
- Whenever increase the sample size, increasing convergence between the estimated values and default values for the shape parameter.
- With all sample sizes under study, increase the default value of the shape parameter θ , increasing MSE and IMSE values relevant to estimate the shape parameter and failure rate function of KD.
- With the large sample size ($n = 50$), increase the default value of the shape parameter θ , decreasing the IMSE values relevant to estimate the reliability function of KD.

In the light of what was stated above, with inverted Levy prior distribution, we recommended the adoption of Bayes estimation method based on square error loss function to estimate the shape parameter, failure rate and reliability functions of KD when the default value of the shape parameter θ is greater than 1.

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