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SOME MULTI-STEP ITERATIVE ALGORITHMS FOR MINIMIZATION OF UNCONSTRAINED NON LINEAR FUNCTIONS

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ABSTRACT

In this paper, we propose new iterative algorithms for minimization of unconstrained non linear functions. Then comparative study is made between new algorithms and Newton's algorithms by means of examples.

Key words: Iterative algorithms; Newton's algorithms; Modified homotopy perturbation method.

1. INTRODUCTION

Optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. In recent times, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques [22] have made deep in to almost all branches of engineering and mathematics. An unconstrained minimization problem is the one where a value of the vector x is sought that minimizes the objective function $f(x)$. This problem can be considered as particular case of the general constrained non-linear programming problem. The study of unconstrained minimization techniques provide the basic understanding necessary for the study of constrained minimization methods and this method can be used to solve certain complex engineering analysis problem. For example, the displacement response (linear or non-linear) of any structure under any specified load condition can be found by minimizing its potential energy.

Vinay Kanwar et.al**.**(2003) introduced new algorithms called, external touch technique and orthogonal intersection technique[23] for solving the non linear equations. Further, they did the comparative study of the new algorithms and Newton's algorithm.

The numerical techniques for solving nonlinear equation have been applied successfully during the last few years. There are many research articles that deal with solving nonlinear equations. The homotopy perturbation method (HPM) was established by He in 1999[12] and systematical description in 2000 [13] which is a coupling of the traditional perturbation method and homotopy in topology [13]. This method was further developed and improved by He and applied to non linear oscillators with discontinuous [14] nonlinear wave equations[15], asymtotology[16], boundary value problem[17], limit cycle and bifurcation of nonlinear functions problems[18] and many other subjects. Thus He's method is a universal one which can solve various kinds of nonlinear equations. Subsequently, many researchers have applied the method to various linear and non linear problems[1, 2, 19, 20]. There has been some development on iterative methods with higher order convergence that required the computation of derivatives of as low order as possible [1-11]. Recently, Arif Rafiq and Muhammad Rafiullah[21] introduced a new article which is some multi-step iterative methods for solving non linear equations, based on modified homotopy perturbation method.

In this paper, we introduce two new algorithms namely New algorithm-I and New algorithm-II for minimization of non linear functions which is based on modified homotopy perturbation method. Then comparative study is made with New algorithms and Newton's algorithm by means of examples.

2. NEW ALGORITHMS

In this section, we introduce two new iterative numerical algorithms for minimization of nonlinear real valued and twice differentiable real functions using the concept of modified homotopy perturbation based numerical algorithms.

Consider the nonlinear optimization problem :

$$\text{Minimize } \{f(x),\ x \in R,\ f: R \to R\}$$

where f is a non-linear twice differentiable function.

2.1. Two step Iterative Algorithms

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by

$$G'(x) = g(x)g''(x)/g'(x).$$

If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

Consider the non linear algebraic equation

$$g(x) = 0, \quad x \in R \qquad \text{---(2.1)}$$

We assume that ω is a simple zero of g(x) and α is an initial guess of sufficiently close to ω. We have the following theorem.

Theorem 2.1: Suppose g is continuous on [a, b] and differentiable in (a, b). Then there exists a point $\delta \in (a,b)$ such that

$$g(y) = g(x) + (y-x)g'(x) + \frac{1}{2}(y-x)^2 g''(\delta) + ...$$

Using the above theorem around α for the equation (2.1), we have

$$g(\alpha) + (x-\alpha)g'(\alpha) + \frac{1}{2}(x-\alpha)^2 g''(\delta) = 0 \qquad \text{----(2.2)}$$

where δ lies between x and α.

"We can rewrite (2.2) in the following form" and the equation (2.2)

$$x = c + N(x) \qquad \text{----(2.3)}$$

where $c = \alpha - \dfrac{g(\alpha)}{g'(\alpha)}$ -----(2.4)

and

$$N(x) = -\frac{1}{2}(x-\alpha)^2 \frac{g''(\delta)}{g'(\alpha)} \qquad \text{-----(2.5)}$$

Proof: Refer the article [21]

We construct a homotopy (mainly due to Golbabai and Javidi [1])

$o : (R \times [0, 1] \times R) \rightarrow R$ for (2.3) which satisfies

$$o(\varpi, \beta, \theta) = \varpi - c - \beta N(\varpi) + \beta(1-\beta)\theta = 0, \ \theta, \ \varpi \in R, \ \beta \in [0, 1] \qquad \text{---(2.6)}$$

where θ is unknown real parameter and β is embedding parameter. It is obvious that

$$o(\varpi, 0, \theta) = \varpi - c = 0 \qquad \text{-----(2.7)}$$

$$o(\varpi, 1, \theta) = \varpi - c - \beta N(\varpi) = 0 \qquad \text{-----(2.8)}$$

The embedding parameter β increases monotonically from zero to unity as the trivial problem. $o(\varpi, 0, \theta) = \varpi - c = 0$ is continuously deformed to the original problem

$o(\varpi, 1, \theta) = \varpi - c - \beta N(\varpi) = 0$.

The modified HPM uses the homotopy parameter β as an expanding parameter to obtain [12]

$$\varpi = x_0 + \beta x_1 + \beta^2 x_2 + ---- \qquad \text{-----(2.9)}$$

The approximate solution of (2.1), therefore, can be readily obtained

$$\varpi = \lim_{\beta \to 1} = x_0 + x_1 + x_2 + ---- \qquad \text{----(2.10)}$$

The convergence of the series (2.10) has been proved by He in his paper [12]. For the application of the modified HPM to (2.1), we can write (2.3) as follows, by expanding $N(\varpi)$ into a Taylor series around x_0:

$$\varpi - c - \beta\left\{ N(x_0) + (\varpi - x_0)\frac{N'(x_0)}{1!} + (\varpi - x_0)^2 \frac{N''(x_0)}{2!} + ...\right\} + \beta(1-\beta)\theta = 0 \qquad \text{---(2.11)}$$

Substituting (2.9) in to (2.11) yields

$$x_0+\beta x_1+\beta^2 x_2+\ldots-c-\beta\begin{pmatrix} N(x_0)+(x_0+\beta x_1+\beta^2 x_2+\ldots-x_0)\dfrac{N'(x_0)}{1!} \\ +(x_0+\beta x_1+\beta^2 x_2+\ldots-x_0)^2\dfrac{N''(x_0)}{2!}+\ldots \end{pmatrix}$$

$$+\beta(1-\beta)\theta=0 \quad ---(2.12)$$

By equating terms with identical powers of β we have,

$$\beta^0:\ x_0=c, \quad ---(2.13)$$

$$\beta^1:\ x_1=N(x_0)-\theta, \quad ---(2.14)$$

$$\beta^2:\ x_2=x_1 N'(x_0)+\theta, \quad ---(2.15)$$

$$\beta^3:\ x_3=x_2 N'(x_0)+\frac{1}{2}x_1^2 N''(x_0) \quad ---(2.16)$$

....

where

$$N(x_0)=-\frac{[g(\alpha)]^2\ g''(\delta)}{2[g'(\alpha)]^3} \quad ----(2.17)$$

$$N'(x_0)=\frac{g(\alpha)\ g''(\delta)}{[g'(\alpha)]^2} \quad ---(2.18)$$

and $$N''(x_0)=-\frac{g''(\delta)}{g'(\alpha)} \quad ---(2.19)$$

Substituting (2.14) into (2.15) and letting $x_2 = 0$, we obtain

$$\theta=\frac{N(x_0)\ N'(x_0)}{N'(x_0)-1} \quad ---(2.20)$$

From (2.4), (2.5) and (2.13) - (2.20) we have,

$$x_0=\alpha-\frac{g(\alpha)}{g'(\alpha)}, \quad --(2.21)$$

$$x_1=-\frac{1}{2}\frac{[g(\alpha)]^2[g''(\delta)]}{[g'(\alpha)]^3-g(\alpha)g'(\alpha)g''(\delta)}, \quad --(2.22)$$

$$x_2=0,$$

$$x_3=-\frac{1}{8[g'(\alpha)]^3}\frac{[g(\alpha)]^4[g''(\delta)]^3}{\left([g'(\alpha)]^2-g(\alpha)g''(\delta)\right)^2}, \quad --(2.23)$$

,
,
,

Substituting (2.21) - (2.24) in (2.10), we can obtain the solution of (2.1) as follows

$$\omega=\alpha-\frac{g(\alpha)}{g'(\alpha)}-\frac{1}{2}\frac{[g(\alpha)]^2[g''(\delta)]}{[g'(\alpha)]^3-g(\alpha)g'(\alpha)g''(\delta)}$$

$$-\frac{1}{8[g'(\alpha)]^3}\frac{[g(\alpha)]^4[g''(\delta)]^3}{\left([g'(\alpha)]^2-g(\alpha)g''(\delta)\right)^2} \quad ----(2.25)$$

This formulation allows us to suggest the following iterative method for solving nonlinear equation (2.1)

Algorithm-1:

For a given x_0, calculate the approximation solution x_{n+1}, by the iterative scheme

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)},\quad g'(x_n)\neq 0 \quad ----(2.26)$$

Algorithm-2:

For a given x_0, calculate the approximation solution x_{n+1}, by the iterative scheme

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}-\frac{1}{2}\frac{[g(x_n)]^2[g''(y_n)]}{[g'(x_n)]^3-g(x_n)g'(x_n)g''(y_n)} \quad ----(2.27)$$

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}, \quad g'(x_n) \neq 0 \qquad \text{---(2.28)}$$

Algorithm-3:

For a given x_0, calculate the approximation solution x_{n+1}, by the iterative scheme

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} - \frac{1}{2}\frac{[g(x_n)]^2[g''(y_n)]}{[g'(x_n)]^3 - g(x_n)g'(x_n)g''(y_n)}$$

$$-\frac{1}{8[g'(x_n)]^3}\frac{[g(x_n)]^4[g''(y_n)]^3}{([g'(x_n)]^2 - g(x_n)g''(y_n))^2} \qquad \text{---(2.29)}$$

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}, \quad g'(x_n) \neq 0 \qquad \text{---(2.30)}$$

Since $g(x) = f'(x)$ the equation (2.27) and (2.28) becomes

New Algorithm – I

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} - \frac{1}{2}\frac{[f'(x_n)]^2[f'''(y_n)]}{[f''(x_n)]^3 - f'(x_n)f''(x_n)f'''(y_n)} \qquad \text{----(2.31)}$$

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}, \quad f''(x_n) \neq 0 \qquad \text{----(2.32)}$$

Since $g(x) = f'(x)$ the equation (2.29) and (2.30) becomes

New Algorithm –II

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} - \frac{1}{2}\frac{[f'(x_n)]^2[f'''(y_n)]}{[f''(x_n)]^3 - f'(x_n)f''(x_n)f'''(y_n)}$$

$$-\frac{1}{8[f''(x_n)]^3}\frac{[f'(x_n)]^4[f'''(y_n)]^3}{([f''(x_n)]^2 - f'(x_n)f'''(y_n))^2} \qquad \text{---(2.33)}$$

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}, \quad f''(x_n) \neq 0 \qquad \text{----(2.34)}$$

3. CONVERGENCE ANALYSIS

Theorem 3.1:

Consider the non linear equation g(x) = 0. Suppose g is sufficiently differentiable. Then for the iterative method defined by Algorithm 2, the convergence is atleast of order 3.

Proof: Refer the article [21]

The same steps in the proof of Theorem3.1 is followed to get the convergence of Algorithm3 which is atleast of order 3.

Refer the article[21]

4.NUMERICAL ILLUSTRATIONS

Example 4.1: Consider the function

$$f(x) = x^3 - 2x - 5, \; x \in R.$$

Then, minimizing point of the function is equal to 0.816467 which is obtained in 3 iterations by Newton's method, in 2 iterations by New algorithm-I and in 2 iterations by New algorithm –II for the initial value x_0 = 1and also seen that no variations in iteration for the initial value of x_0 = 2 and x_0 = 3.

Iterations	Newton's Algorithm	New Algorithm- I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.833333	0.833333	0.833333
2	0.816667	0.816497	0.816497
3	0.816497		
4.			

Iterations	Newton's Algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.166667	1.166667	1.166667
2	0.869048	0.869048	0.869048
3	0.818085	0.818085	0.818085
4.	0.816498	0.816498	0.816498
5.	0.816497	0.816497	0.816497

Iterations	Newton's Algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	1.611111	1.611111	1.611111
2	1.012452	1.012452	1.012452
3	0.835460	0.835460	0.835460
4.	0.816712	0.816712	0.816712
5.	0.816497	0.816497	0.816497

Example 4.2: Consider the function

$$f(x) = x^4 - x - 10 \quad x \in R .$$

Then, minimizing point of function is equal to 0.629961 which is obtained in 4 iterations by Newton's Algorithm, in 4 iterations by New algorithm-I and in 4 iterations by New algorithm-II for the initial value $x_0 = 1$ and also seen that no variation of iterations for the initial value of $x_0 = 2$ and $x_0 = 3$.

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.750000	0.750000	0.750000
2	0.648148	0.648148	0.648148
3	0.630466	0.630466	0.630466
4	0.629961	0.629961	0.629961

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.354167	1.354167	1.354167
2	0.948222	0.948222	0.948222
3	0.724830	0.724830	0.724830
4	0.641836	0.641836	0.641836
5	0.630179	0.630179	0.630179
6	0.629961	0.629961	0.629961

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	2.009259	2.009259	2.009259
2	1.360148	1.360148	1.360148
3	0.951810	0.951810	0.951810
4	0.726525	0.726525	0.726525
5	0.642227	0.642227	0.642227
6	0.630193	0.630193	0.630193
7	0.629961	0.629961	0.629961

Example 4.3: Consider the function

$$f(x) = xe^x - 1 \,,\; x \in R \,.$$

Then, minimizing point of the function is equal to -1 which is obtained in 7 iterations by Newton's algorithm, 7 iterations by New algorithm-I and in 7 iterations by New algorithm-II for the initial value $x_0 = 1$ and also seen that no variations in iterations for the initial value of $x_0 = 2$ and $x_0 = 3$.

Iterations	Newton's algorithm	New Algorithm- I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.333333	0.333333	0.333333
2	-0.238095	-0.238095	-0.238095
3	-0.670528	-0.670528	-0.670528
4	-0.918350	-0.918350	-0.918350
5	-0.993836	-0.993836	-0.993836
6	-0.999962	-0.999962	-0.999962
7	-1	-1	-1

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.250000	1.250000	1.250000
2	0.557692	0.557692	0.557692
3	-0.051330	-0.051330	-0.051330
4	-0.538159	-0.538159	-0.538159
5	-0.854409	-0.854409	-0.854409
6	-0.981421	-0.981421	-0.981421
7	-0.999661	-0.999661	-0.999661
8	-0.999999	-0.999999	-0.999999
9	-1.000000	-1.000000	-1.000000

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	2.200000	2.200000	2.200000
2	1.438095	1.438095	1.438095
3	0.728954	0.728954	0.728954
4	0.095395	0.095395	0.095395
5	-0.427368	-0.427368	-0.427368
6	-0.791491	-0.791491	-0.791491
7	-0.964025	-0.964025	-0.964025
8	-0.998751	-0.998751	-0.998751
9	-0.999999	-0.999999	-0.999999
10	-1.000000	-1.000000	-1.000000

5.CONCLUSION

In this paper, we propose two new iterative algorithms namely New algorithm – I and New algorithm – II for minimization of unconstrained nonlinear functions and then the comparative study is made with Newton's Algorithm. It is clear from the numerical results that the rate of convergence of new algorithms is almost equal to Newton's Algorithm. By the numerical results, we have the rate of convergence of New algorithms and Newton's algorithms are almost all same. So, we have alternative algorithms for Newton's algorithm for minimization of unconstrained non linear functions is established. In future, we may extend the new algorithms for constrained optimization problems

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