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STABILITY ANALYSIS OF PREY-PREDATOR AND SCAVENGER MODEL WITH TWO FUNCTIONAL RESPONSE AND LINEAR HARVESTING

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ABSTRACT
In this paper a mathematical model prey-predator and scavenger with linear harvesting in predator and scavenger. The existence, uniqueness, limitations (roundedness) of solution and the stability analysis of every possible accumulation points are studied. The Lyapunov function is used to study the global dynamics of the model. The effect of the scavenger and harvest on system dynamics is discussed using numerical simulation.

KEYWORDS: prey-predator, scavenger, Holling type III functional response, linear harvesting

1. INTRODUCTION

In recent years, it has been important to study dynamics of impulsive disorders on population models. In particular, the models of impulsive assemblies of prey and predator were verified by many researchers, Then these studies began to expand more and more , Anderson and May [1,2] were the first to combine ecology and epidemiology with Lotka-Volterra prey and predator model with infection disease extend among prey by contact between them and no reproduction in infected prey, where the dynamics of the epidemiological ecosystem and harvesting systems were studied, and the latter is considered one of the most important factors that help stop diseases and transmission of infection Where most interactions have been described with response functions known as functions Holing and are of several types[3,4,5]:

1- $y = ax$ linear type
2- $y = \frac{ax}{b+x}$ Holing –II
3- $y = \frac{ax^n}{b+x^n}$ Holling –III

Many researchers have studied the dynamics of prey and predator systems using the Holling response functions, including the first type and the second type R.K. Naji [6], and the third type pan-ping Liu [7].
Later, the prey -predator model was studied in the presence of a scavenger, since it is known that scavengers are animals that eat dead (cadaver) organisms that have died or have been killed by other predatory organisms While raking generally survey of raptors to carnivores that feed on carrion, it is also a herbivorous feeding behavior scavengers play an important role in the ecosystem by consuming dead animal and plant material. Analyzers and catalysts complete this process, by consuming residue sweepers points Decomposers and detritivores complete this process, by consuming the remains scavengers points. There is an influence on the dynamics of the prey and predator system, a small number of researchers have studied the effect of scavenger presence in their models. Nolting et al. [8] proposed and analyzed a three species system consisting of a predator, its prey and a scavenger and studied R.K.Naji and H.A.Satar prey-predator and scavenger model [9]
Further studied O.A.Ali and A.A.Majeed The dynamics of prey-predator model with harvesting involving diseases in both populations[10] , and the More of researcher are studied prey-predator model with nonlinear harvesting O.M.Ali studied the SIR model with non-linear harvesting and vaccination[11] and R.K.Naji and H.A.Satar studied prey-predator model with non-linear harvesting [12].
In accordance with the above, in this paper, however, we proposed and analyzed a food web model consisting of three species prey–predator–scavenger with two types of functional response to predation and a linear harvesting function in predator and scavenger. Our aim is to study the effect of harvesting and maximum attack rate of scavenger on the dynamics of the system.

2.1 Mathematical model

In this part an ecological system consisting of prey–predator–scavenger if formulated mathematically for study. It's assumed that there is a linear harvesting on predator and scavenger. So in order to formulate the dynamics of such a real life system the following hypotheses are adopted**:**

1- Let $X(T)$, $Y(T)$ and $Z(T)$ represent the densities at time t for the prey, predator and scavenger, respectively. It's assumed that prey species grow logistically with intrinsic growth rate $r > 0$ and carrying capacity $K > 0$.

2- predator consumes prey according to Lotka-Volterra of functional response with maximum attack rate $c > 0$, while the scavenger consume prey according to Holing- III types of functional response with maximum attack rate $a > 0$, and half saturation rate $b_1 > 0$.

3- The predator, in the absence of prey, it decomposes exponentially with death rate $d_1 > 0$, however the existence of the prey contributes to predator's growth with conversion rate $e_1 > 0$.

4- The scavenger, in the absence of its prey and all other resources of food, it significantly degrades with a natural mortality rate $d_2 > 0$, however the existence of their prey contributes to scavenger's growth with conversion rate $e_2 > 0$. In addition it's assumed that the population of scavenger benefits from naturally died predator and with benefit rate $b_2 > 0$

5- Finally, predator and scavenger assemblies are assumed to be harvested by external forces according to linear type of harvesting function with constant rates H_1 >0 and H_2>0 for the predator and scavenger respectively

According to the above hypotheses the dynamics of the above food web system can be described mathematically with the following set of first order ordinary differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{k}\right) - cXY - \frac{aX^2Z}{b_1 + X^2} \\
\frac{dY}{dT} &= e_1\, cXY - d_1Y - H_1Y \qquad (2.1)\\
\frac{dX}{dT} &= e_2\frac{aX^2Z}{b_1+X^2} + b_2YZ - H_2Z - d_2Z
\end{aligned}$$

Note that the model proposed above contains (11) parameters that make the mathematical analysis of the system difficult. So in order to reduce the number of parameters and determine which parameter represents the control parameter, the following dimensionless variables are used:

$$t = r\,T\,, x = \frac{X}{k}\ \ , y = \frac{c\,Y}{r}\ \ , z = \frac{aZ}{rk}.$$

Then system (2.1) can be written in the following dimensionless form:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(1-x-y-\frac{x^2z}{b+x^2}\right) = f_1(x,y,z) \\
\frac{dy}{dt} &= y(\gamma x - \delta - u) = f_2(x,y,z) \qquad (2.2)\\
\frac{dz}{dt} &= z\left(\frac{\alpha x^2}{b+x^2} + \theta\, y - E - F\right) = f_3(x,y,z)
\end{aligned}$$

Where

$$b = \frac{b_1}{k}\,, \gamma = \frac{e_1ck}{r}\,, \delta = \frac{d_1}{r}\,, \mu = \frac{H_1}{r}\,, \alpha = \frac{e_2a}{r},$$
$$F = \frac{d_2}{r}\,, \theta = \frac{b_2}{c}\,, E = \frac{H_2}{r}$$

With $x(0) \geq 0$,$y(0) \geq 0$ and $z(0) \geq 0$.

represent the dimensionless parameter of system (2.2). It is observed that the number of parameters have been reduced from eleven in the system (2.1) to eight in the system (2.2).

It is easy to check that all the interaction functions f_1, $f_2$and f_3 on the right side of system (2.2) are continuous and have continuous partial derivatives on R_+^3 with respect to dependent variables x , y and z. Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Furthermore the boundedness of the system is shown in the following theorem.

Theorem (2.1): All the solutions of system (2.2) which initiate in R_+^3 are uniformly bounded.

Proof.

Assume that $(x(t), y(t), z(t))$ be any solution of the system (2.2) with non-negative initial condition $(x(0), y(0), z(0))$. According to the first equation of system (2.2) we have:

$$\frac{dx}{dt} \leq x(1-x)$$

Clearly according to the theory of differential inequality, we get:

$\limsup_{t\to\infty} x(t) \leq 1$. Define the function

$$M(t) = x(t) + \frac{y(t)}{\gamma} + \frac{z(t)}{\alpha}.$$

Therefore,

$$\frac{dM}{dt} < x(1-x) - \frac{y}{\gamma}(\delta+\mu) - \frac{z}{\alpha}(E+F-\theta y)$$

$$\frac{dM}{dt} \leq 2 - nM \qquad \text{where } n = \min\{\,\delta+\mu, E+F-\theta\sup(y)\,\}.$$

$$M(t) \leq \frac{2}{n} + \left(M(0) - \frac{2}{n}\right)e^{-nt} \quad .$$

Thus $0 \leq M(t) \leq \frac{2}{n}$ as $t \to \infty$. Hence all the solutions of system (2.2) are uniformly bounded and the proof is complete

2.2 Existence of accumulation points

In this section, the existence of every possible accumulation points of the system (2.2) is discussed. it is observed that , system (2.2) has at most seven accumulation points.

1) The vanishing accumulation point $E_0 = (0\,,0\,,0\,)$ always exists.
2) The axial accumulation point $E_1 = (1\,,0\,,0\,)$ always exists.
3) The scavenger-free accumulation point $E_2 = (\bar{x}\,,\bar{y}\,,0\,)$ exists iff there is a positive solution to the following set of equations:

$$1 - x - y = 0 \qquad (2.2a)$$
$$\gamma x - \delta - \mu = 0 \qquad (2.2b)$$

From the equation $(2.2b)$ we have,

$$\bar{x} = \frac{\delta+\mu}{\gamma} \quad (2.2c)$$

Now, by substituting equation (2.2c) in equation $(2.2a)$ we get:

$$\bar{y} = 1 - \frac{\delta+\mu}{\gamma} \quad (2.2d)$$

Note that the equation $(2.2d)$ is a positive, provided that:

$$\gamma > \delta + \mu \quad (2.2e)$$

$$\frac{\alpha x^2}{b+x^2} - E - F = 0 \ \rightarrow (\alpha - (F+E))x^2 - b = 0 \quad (2.3b)$$

From equation $(2.3b)$ we

$$h\dot{x} = \frac{\sqrt{b(F+E)}}{(\alpha-(F+E))} \quad (2.3c)$$

Note that the equation $(2.3c)$ is a positive root, provided that:

$$\alpha > F + E \quad (2.3d)$$

Now, by substituting equation (2.3c) in equation $(2.3a)$ we get:

$$\dot{z} = \frac{(1-\dot{x})(b+\dot{x}^2)}{\dot{x}} \quad (2.3e)$$

Note that the equation $(2.3e)$ is a positive, provided that:

$$\alpha > F + E + \sqrt{b(F+E)} \quad (2.3f)$$

5) The positive accumulation point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$ exists and unique in the Int. R_+^3 of xyz- space iff there is a positive solution to the following set of equations:

$$1 - x - y - \frac{xz}{b+x^2} = 0 \quad (2.4a)$$

$$\gamma x - \delta - \mu = 0 \quad (2.4b)$$

$$\alpha\frac{x^2}{b+x^2} + \theta y - F - E = 0 \quad (2.4c)$$

From equation (2.4b) we have,

$$\bar{\bar{x}} = \frac{\delta+\mu}{\gamma}, \quad (2.4d)$$

4) The predator-free accumulation point $E_3 = (\dot{x}, 0, \dot{z})$ exists iff there is a positive solution to the following set of equations:

$$1 - x - \frac{xz}{b+x^2} = 0 \quad (2.3a)$$

Now, by substituting the equations (2.4d) in the equation (2.4c) we get:

$$\bar{\bar{y}} = \frac{b(F+E) + (F+E-\alpha)\left(\frac{\delta+\mu}{\gamma}\right)^2}{\theta}, \quad (2.4e)$$

Note that the equation $(2.4e)$ is a positive root, provided that:

$$F + E > \alpha \quad (2.4f)$$

Now, by substituting equations (2.4e) and (2.4d) in equation (2.4a) we get:

$$\bar{\bar{z}} = \left(1 - \left(\frac{\delta+\mu}{\gamma}\right) - \left(\frac{b(F+E)+(F+E-\alpha)\left(\frac{\delta+\mu}{\gamma}\right)^2}{\theta}\right)\right) * \frac{\left(b+\left(\frac{\delta+\mu}{\gamma}\right)^2\right)}{\left(\frac{\delta+\mu}{\gamma}\right)}, \quad (2.4g)$$

Note that the equation $(2.4g)$ is a positive root, provided that:

$$1 > \bar{\bar{x}} + \bar{\bar{y}} \quad (2.4h)$$

Consequently, the positive accumulation point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$ of system (2.2) exists uniquely in the $Int. R_+^3$ of xyz −space .

2.3 Local Stability Analysis.

In this part, we analyzed the local stability of the model (2.2) around all accumulation point and discussed through computing the Jacobian matrix $J(x, y, z)$ and determined the eigenvalues of system (2.2) at each of them the Jacobian matrix $J(x,y,z)$ of the system (2.2) at all of them can be written:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 - 2x - y - \left(\frac{2bxz}{(b+x^2)^2}\right) & -x & -\frac{x^2}{b+x^2} \\ \gamma x & \gamma x - \delta - \mu & 0 \\ \frac{2\alpha bxz}{(b+x^2)^2} & \theta z & \frac{\alpha x^2}{b+x^2} + \theta y - F - E \end{pmatrix}. \quad (2.7)$$

- **Stability of the accumulation point $E_0 = (0, 0, 0)$**
  The Jacobian matrix of system (2.2) at E_0 can be written as,

$$J_0 = J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\delta-\mu & 0 \\ 0 & 0 & -F-E \end{bmatrix} \quad (2.8)$$

Then the distinctive equation of $J(E_0)$ is given by:

$$(1-\lambda)\,(-\delta-\mu-\lambda)(-F-E-\lambda) = 0,$$

So, the eigenvalues of J_0 are

$$\lambda_p = 1\,, \lambda_{pr} = -\delta-\mu, \lambda_s = -F-E$$

Thus, the accumulation point E_0 is saddle point

- **Stability of the accumulation point $E_1 = (1,0,0)$**
The Jacobian matrix of system (2.2) at E_1 can be written as,

$$J_1 = J(E_1) = \begin{bmatrix} -1 & -1 & -\frac{1}{b+1} \\ \gamma & \gamma-\delta-\mu & 0 \\ 0 & 0 & \frac{\alpha}{1+b} - F - E \end{bmatrix} \quad (2.9a)$$

Then the distinctive equation of $J(E_1)$ is given by:

$$[\lambda^2 + B_1\lambda + B_2]\left(\frac{\alpha}{1+b} - F - E - \lambda\right) = 0,$$

where:

$B_1 = (\delta + \mu + 1 - \gamma) > 0$

$B_2 = \delta + \mu > 0$

So, either

$$\left(\frac{\alpha}{1+b} - F - E - \lambda\right) = 0, \quad (2.9b)$$

which gives two eigenvalues of $J(E_1)$ by:

$\lambda_s = \frac{\alpha}{1+b} - F - E$, which is negative if the following conditions hold.

$$\frac{\alpha}{1+b} < F + E \quad (2.9c)$$

Or

$$\lambda^2 + B_1\lambda + B_2 = 0$$

which gives that other two eigenvalues of J_1 with negative real parts which are (by using Routh Hurwitz criteria)

$$\lambda_p = \frac{1}{2}\left(-B_1 + \sqrt{B_1^2 - 4B_2}\right),$$

$$\lambda_{pr} = \frac{1}{2}\left(-B_1 - \sqrt{B_1^2 - 4B_2}\right).$$

So, accumulation point E_1 is locally asymptotically stable in the $.R_+^3$. However, it is unstable otherwise.

- **Stability of the accumulation point $E_2 = (\bar{x}, \bar{y}, 0)$**
The Jacobian matrix of system (2.2) at E_2 can be written as,

$$J_2 = J(E_2) = \begin{bmatrix} -\left(\frac{\delta+\mu}{\gamma}\right) & -\left(\frac{\delta+\mu}{\gamma}\right) & -\frac{(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} \\ \delta+\mu & 0 & 0 \\ 0 & 0 & \frac{\alpha\,(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} + \theta(1 - \left(\frac{\delta+\mu}{\gamma}\right)) - F - E \end{bmatrix} \quad (2.10)$$

Then the distinctive equation of $J(E_2)$ is given by

$$\left[\lambda^2 + \left(\frac{\delta+\mu}{\gamma}\right)\lambda + \left(\frac{(\delta+\mu)^2}{\gamma}\right)\right]\left(\frac{\alpha\,(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} + \theta(1 - \left(\frac{\delta+\mu}{\gamma}\right)) - F - E - \lambda\right) = 0,$$

So, either

$$\left(\frac{\alpha\,(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} + \theta(1 - \left(\frac{\delta+\mu}{\gamma}\right)) - F - E - \lambda\right) = 0, \quad (2.10a)$$

which gives eigenvalues of $J(E_2)$ by:

$$\lambda_s = \frac{\alpha\,(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} + \theta(1 - \left(\frac{\delta+\mu}{\gamma}\right)) - F - E,$$

which is negative if the following conditions hold.

$$F + E > \frac{\alpha\,(\delta+\gamma)^2}{b\gamma^2 + (\delta+\gamma)^2} + \theta\left(1 - \left(\frac{\delta+\mu}{\gamma}\right)\right) \quad (2.10b)$$

Or

$$\left[\lambda^2 + \left(\frac{\delta+\mu}{\gamma}\right)\lambda + \left(\frac{(\delta+\mu)^2}{\gamma}\right)\right] = 0$$

which gives that other two eigenvalues of J_2 with negative real parts which are (by using Routh Hurwitz criteria)

$$\lambda_p = \frac{1}{2}\left(-\left(\frac{\delta+\mu}{\gamma}\right) + \sqrt{\left(\left(\frac{\delta+\mu}{\gamma}\right)\right)^2 - 4\left(\frac{(\delta+\mu)^2}{\gamma}\right)}\right),$$

$$\lambda_{pr} = \frac{1}{2}\left(-\left(\frac{\delta+\mu}{\gamma}\right) - \sqrt{\left(\left(\frac{\delta+\mu}{\gamma}\right)\right)^2 - 4\left(\frac{(\delta+\mu)^2}{\gamma}\right)}\right).$$

So, accumulation point E_2 is locally asymptotically stable in the $.R_+^3$. However, it is unstable otherwise.

- **Stability of the accumulation point $E_3 = (\dot{x}, 0, \dot{z})$**
The Jacobian matrix of system (2.2) at E_3 can be written as,

$$J_3 = J(E_3) = |z_{ij}]_{3\times3}, \quad (2.11)$$

where

$z_{11} = \left(1 - 2\dot{x} - \frac{-2b\dot{x}\dot{z}}{(b+\dot{x})^2}\right), z_{12} = -\dot{x},\ z_{13} = -\frac{\dot{x^2}}{c_3 + \dot{x^2}},$

$z_{21} = \gamma\dot{x},\ z_{22} = \gamma\dot{x} - \delta - \mu\,, z_{23} = 0,$

$z_{31} = \frac{2\alpha b\dot{x}\dot{z}}{\left(b+\dot{x^2}\right)^2}, z_{32} = \theta\dot{z}\,, \qquad z_{33} = 0,$

Then the distinctive equation of $J(E_3)$ is given by:

$$[\,\lambda^3 + V_1\,\lambda^2 + V_2\lambda + V_3\,] = 0, \qquad (2.11a)$$

where:

$V_1 = -(z_{11} + z_{22})$
$V_2 = z_{11}z_{22} - z_{12}z_{21} - z_{13}z_{31}$
$V_3 = z_{13}z_{31}z_{22} - z_{13}z_{21}z_{32}$

Using Routh Hurwitz criterion implies equation $(2.11a)$ has roots (eigenvalues)with negative real parts if and only if $V_1 > 0, V_3 > 0$ and $V_1V_2 - V_3 > 0$.
$V_1 = -(z_{11} + z_{22}) > 0$
$V_2 = z_{11}z_{22} - z_{12}z_{21} - z_{13}z_{31}$
$V_3 = z_{13}z_{31}z_{22} - z_{13}z_{21}z_{32} > 0$, provided that
$$\gamma\dot{x} < \delta + \mu, \qquad (2.11b)$$

Further, it is easy to check that:

$$V_1V_2 - V_3 = (z_{11} + z_{22})(-z_{11}z_{22} + z_{12}z_{21} + z_{13}z_{31}) - z_{13}z_{31}z_{22} + z_{13}z_{21}z_{32}$$
$$= (z_{11} + z_{22})(-z_{11}z_{22} + z_{12}z_{21}) + z_{11}z_{13}z_{31} + z_{13}z_{21}z_{32} > 0$$

provided that
$(z_{11} + z_{22})(-z_{11}z_{22} + z_{12}z_{21}) + z_{11}z_{13}z_{31} > z_{13}z_{21}z_{32}$

The accumulation point E_3 is locally asymptotically stable in the $.R_+^3$. However, it is unstable otherwise.

- **Stability of the accumulation point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$**

The Jacobian matrix of system (2.2) at E_4 can be written as,

$$J_4 = J(E_4) = \left[d_{ij}\right]_{3\times 3}, \qquad (2.12)$$

Where

$d_{11} = (1 - 2\bar{\bar{x}} - \bar{\bar{y}} - \frac{2b\bar{\bar{x}}\bar{\bar{z}}}{\left(b+\bar{\bar{x}}^2\right)^2}), d_{12} = -\bar{\bar{x}}, d_{13} = -\frac{\overline{\overline{x^2}}}{b+\overline{\overline{x^2}}},$

$d_{21} = \gamma\bar{\bar{x}}, \qquad d_{22} = 0,\ d_{23} = 0,\ d_{31} = \frac{2\alpha b\bar{\bar{x}}\bar{\bar{z}}}{\left(b+\bar{\bar{x}}^2\right)^2}, d_{32} = \theta\bar{\bar{z}}\,, d_{33} = 0\,,$

Then the distinctive equation of $J(E_4)$ is given by:

$$[\,\lambda^3 + U_1\,\lambda^2 + U_2\lambda + U_3\,] = 0, \qquad (2.12a)$$

where
$U_1 = -d_{11}$
$U_2 = -(d_{12}d_{21} + d_{13}d_{31})$
$U_3 = -(d_{13}d_{21}d_{32})$

Using Routh Hurwitz criterion implies equation $(2.12d)$ has roots (eigenvalues)with negative real parts if and only if $U_1 > 0, U_3 > 0$ and $U_1U_2 - U_3 > 0$.

Now $U_1 > 0$, provided that

$$2\bar{\bar{x}} + \bar{\bar{y}} + \frac{2b\bar{\bar{x}}\bar{\bar{z}}}{\left(b+\bar{\bar{x}}^2\right)^2} > 1 \qquad (2.12e)$$

Also, we obtain that $U_3 > 0$

Further, it is easy to check that:

$U_1U_2 - U_3 = (d_{11}d_{12}d_{21} + d_{13}d_{31}d_{11}) - d_{13}d_{21}d_{32} > 0$
,
provided that :

$$U_1U_2 > 0 \qquad (2.12g)$$

Therefore , each the eigenvalues of $J(E_4)$ have a negative real part under the given conditions , so E_4 is locally asymptotically stable. However, it is unstable otherwise.

2.4 Global stability analysis

In this part the global stability analysis for the accumulation points, which are locally stable asymptotically to the system (2.2) is studied analytically using a appropriate Lyapunov function as shown in the following theorems.

Theorem (5.1)

Assume that the predator and scavenger free accumulation point $E_1 = (1\,,0\,,0\,)$ of system $(\,2.2\,)$ is locally asymptotically stable in the R_+^3.Then E_1 is globally asymptotically stable on the sub region $\omega_1 \subseteq R_+^3$ provided that the following condition hold:

$$\delta + \mu > \gamma \qquad (2.15a)$$

$$G_1(x,y,z,w) = (\,x - 1 - \,ln\,x\,) + \frac{y}{\gamma} + \frac{z}{\alpha}$$

It is easy to see that $G_1(x,y,z,w) \epsilon\, C^1(R_+^3, R)$, and $G_1(E_1) = 0$, and $G_1(x,y,z) > 0\,; \forall (x,y,z) \neq E_1$. Now by differentiating G_1 with respect to time t and carrying some algebraic handling, given that:

$$\frac{dG_1}{dt} = -(x-1)^2 - \frac{y}{\gamma}\left(\frac{\delta+\mu}{\gamma} - 1\right) - \frac{z}{\alpha}\left(E + F - \theta y - \frac{\alpha x}{b+x^2}\right)$$

$$\frac{dG_1}{dt} < -(x-1)^2 - y\left(\frac{\delta+\mu}{\gamma} - 1\right) - z(E + F - \theta y - \alpha x)$$

Thus, $\frac{dG_1}{dt}$ is negative definite and hence G_1 is Lyapunov function under the condition $(2.15a)$ and$(2.15b)$. So E_1 is a globally asymptotically stable on the sub region $\omega_1 \subseteq R_+^3$ and then the proof is complete

Theorem (5.2) :

Assume that the scavenger free accumulation point $E_2 = (\bar{x}, \bar{y}, 0)$ of system (2.2) is locally asymptotically stable in the R_+^3.Then E_2 is globally asymptotically stable on the sub region $\omega_2 \subseteq R_+^3$ provided that the following conditions hold:

$$F + E > \frac{\theta}{\alpha}\max(y) - \alpha\bar{x}\max(x) \qquad (2.16a)$$

Proof: Consider the following function

$$G_2(x,y,z) = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + \frac{y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}}{\gamma} + \frac{z}{\alpha}$$

It is easy to see that $G_2(x,y,z,w) \epsilon C^1(R_+^3, R)$, and $G_2(E_2) = 0$, and $G_2(x,y,z) > 0 ; \forall (x,y,z) \neq E_2$. Now by differentiating G_2 with respect to time t and going some algebraic handling, given that:

$$E + F > \theta\max(y) + \alpha\max(x) \qquad (2.15b)$$

Proof: Consider the following function

$$\frac{dG_2}{dt} = -(x-\bar{x})^2 - (x-\bar{x})\frac{xz}{b+x^2} + \frac{x^2 z}{b+x^2} + \frac{\theta}{\alpha}yz - \frac{(F+E)}{\alpha}z$$

$$\frac{dG_2}{dt} < -(x-\bar{x})^2 - \frac{z}{\alpha}(F + E - \frac{\theta}{\alpha}y - \alpha\bar{x}x)$$

Thus, $\frac{dG_2}{dt}$ is negative definite and hence G_2 is Lyapunov function under the condition $(2.16a)$ So E_2 is a globally asymptotically stable on the sub region $\omega_2 \subseteq R_+^3$and then the proof is complete

Theorem (5.3)

Assume that the predator free accumulation point $E_3 = (\dot{x}, 0, \dot{z})$ of system (2.2) is locally asymptotically stable in the R_+^3.Then E_3 is globally asymptotically stable on the sub region $\omega_3 \subseteq R_+^3$ provided that the following conditions hold:

$$\beta_1 > \beta_2 \qquad (2.17a)$$

$$z > \dot{z} \qquad (2.17b)$$

$$\frac{\delta+\mu}{\gamma} > \dot{x} \qquad (2.17c)$$

where

$$\beta_1 = (x-\dot{x})^2 + y\left(\frac{\delta+\mu}{\gamma} - \dot{x}\right) + \frac{x^2\dot{z}}{b+x^2} + \frac{z\dot{x}^2}{b+1}$$

And

$$\dot{\beta}_2 = (z-\dot{z})\frac{\theta}{\alpha}y + \frac{x\dot{x}\dot{z}}{b+\dot{x}^2} + \frac{\dot{x}xz}{b+x^2}$$

Proof: Consider the following function

$$G_3(x,y,z,w) = \left(x - \dot{x} - \dot{x}\ln\frac{x}{\dot{x}}\right) + \frac{y}{\gamma} + \frac{z - \dot{z} - \dot{z}\ln\frac{z}{\dot{z}}}{\alpha}$$

It is easy to see that $G_3(x,y,z,w) \epsilon C^1(R_+^3, R)$, and $G_3(E_3) = 0$, and $G_3(x,y,z) > 0 ; \forall (x,y,z) \neq E_3$. Now by differentiating G_3 with respect to time t and going some algebraic handling ,given that

$$\frac{dG_3}{dt} = -(x-\dot{x})^2 - y\left(\frac{\delta+\mu}{\gamma} - \dot{x}\right) + (x-\dot{x})\left(\frac{\dot{x}\dot{z}}{b+\dot{x}^2} - \frac{xz}{b+x^2}\right) + (z-\dot{z})\left(\frac{x^2}{b+x^2} + \frac{\theta}{\alpha}y - \frac{\dot{x}^2}{b+\dot{x}^2}\right)$$

$$\frac{dG_3}{dt} < -(x-\dot{x})^2 - y\left(\frac{\delta+\mu}{\gamma} - \dot{x}\right) + (z-\dot{z})\frac{\theta}{\alpha}y + \frac{x\dot{x}\dot{z}}{b+\dot{x}^2} + \frac{\dot{x}xz}{b+x^2} - \frac{x^2\dot{z}}{b+x^2} - \frac{z\dot{x}^2}{b+1} = -\beta_1 + \beta_2$$

Thus, $\frac{dG_3}{dt}$ is negative definite and hence G_3 is Lyapunov function under the conditions $(2.17a) - (2.17c) and\ (2.11c)$. So E_3 is a globally asymptotically stable on the sub region $\omega_3 \subseteq R_+^3$ and then the proof is complete ■

Theorem (5.4) :

Assume that the infected predator free accumulation point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}},)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_4 is globally asymptotically stable on the sub region $\omega_4 \subseteq R_+^3$ provided that the following conditions hold:

$$\bar{\bar{\beta}}_1 > \bar{\bar{\beta}}_2 \qquad (2.18a)$$

$$x > \bar{\bar{x}}, y > \bar{\bar{y}}, z > \bar{\bar{z}} \qquad (2.18b)$$

where

$$\bar{\bar{\beta}}_1 = (x-\bar{\bar{x}})^2 + (x-\bar{\bar{x}})\left(\frac{xz(b+\bar{\bar{x}}^2)}{b(b+\bar{\bar{x}}^2)}\right),$$

$$and$$

$$\bar{\bar{\beta}}_2 = (x-\bar{\bar{x}})\left(\frac{\bar{\bar{x}}\bar{\bar{z}}(b+x^2)}{(b+x^2)(b+\bar{\bar{x}}^2)}\right)+\frac{\theta}{\alpha}(y-\bar{\bar{y}})(z-\bar{\bar{z}})+(z-\bar{\bar{z}})\left(\frac{b(x^2-\bar{\bar{x}}^2)}{(b+x^2)(b+\bar{\bar{x}}^2)}\right)$$

Proof: Consider the following function

$$G_4(x,y,z,w)=\left(x-\bar{\bar{x}}-\bar{\bar{x}}\,ln\frac{x}{\bar{\bar{x}}}\right)+\frac{\left(y-\bar{\bar{y}}-\bar{\bar{y}}ln\frac{y}{\bar{\bar{y}}}\right)}{\gamma}+\frac{z-\bar{\bar{z}}-\bar{\bar{z}}ln\frac{z}{\bar{\bar{z}}}}{\alpha}$$

It is easy to see that $G_4(x,y,z)\epsilon\, C^1(R_+^3,R)$, and $G_4(E_4)=0$, and $G_4(x,y,z)>0;\ \forall(x,y,z)\neq E_4$. : Now by differentiating G_4 with respect to time t and going some algebraic handling, given that

$$\frac{dG_4}{dt}=-(x-\bar{\bar{x}})^2-(x-\bar{\bar{x}})\left(\frac{xz(b+\bar{\bar{x}}^2)-\bar{\bar{x}}\bar{\bar{z}}(b+x^2)}{(b+x^2)(b+\bar{\bar{x}}^2)}\right)+\frac{\theta}{\alpha}(y-\bar{\bar{y}})(z-\bar{\bar{z}})+(z-\bar{\bar{z}})\left(\frac{b(x^2-\bar{\bar{x}}^2)}{(b+x^2)(b+\bar{\bar{x}}^2)}\right)$$

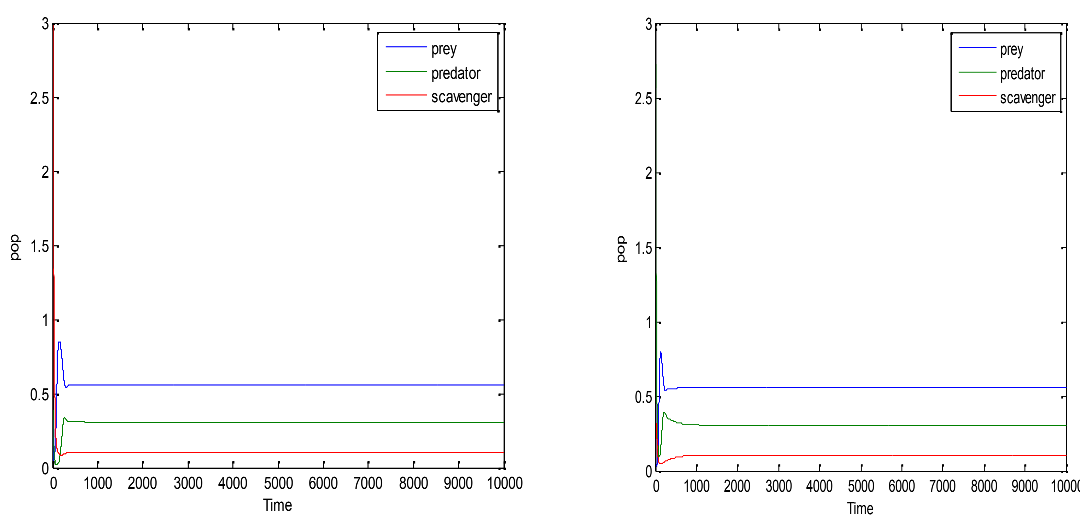
$$\frac{dG_4}{dt}<-(x-\bar{\bar{x}})^2-(x-\bar{\bar{x}})\left(\frac{xz(b+\bar{\bar{x}}^2)-\bar{\bar{x}}\bar{\bar{z}}(b+x^2)}{b(b+\bar{\bar{x}}^2)}\right)+\frac{\theta}{\alpha}(y-\bar{\bar{y}})(z-\bar{\bar{z}})+(z-\bar{\bar{z}})\left(\frac{b(x^2-\bar{\bar{x}}^2)}{b(b+\bar{\bar{x}}^2)}\right)=-\bar{\bar{\beta}}_1+\bar{\bar{\beta}}_2$$

Thus, $\frac{dG_4}{dt}$ is negative definite and hence G_4 is Lyapunov function under the conditions $(2.18a),(2.18b)$. So E_4 is a globally asymptotically stable on the sub region $\omega_4\subseteq R_+^3$ and then the proof is complete

6. Numerical simulation

In this part, we confirmed our obtained results in the previous parts numerically by using Runge Kutta method along with predictor corrector method. Note that, we use turbo C++ in programming and matlab in plotting and then discuss our obtained results. The system (2.2) is studied numerically for different sets of parameters and different sets of strating points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2.2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters:

$$\boldsymbol{b=0.1,\gamma=0.9,\delta=0.2,\mu=0.3,\alpha=0.5,\theta=0.4,E=0.2,F=0.3,}\quad (\boldsymbol{2.5a})$$



Clearly, figure (1) shows that system (2.2) approaches asymptotically to the positive accumulation point $E_4 = □$ (0. 5, 0.3, 0.1) starting from two starting point and this is confirming our obtained analytical results

Now, in order to discuss the effect of the parameters values of system (2.2) on the dynamical behavior of the system, the system is solved numerically for the data given in (2.5a) with varying one parameter at each time and sometime two parameters the obtained results are given below.

The saturation rate in the range 0. 2 < b < 0.9 ,keeping other parameters as data given in (2.5a) ,causes extinction in the scavenger and the system will approach to the scavenger free accumulation point. However for 0.1 < b <

0.2 , it is observed that system (2.2) still approach asymptotically to the positive accumulation point.

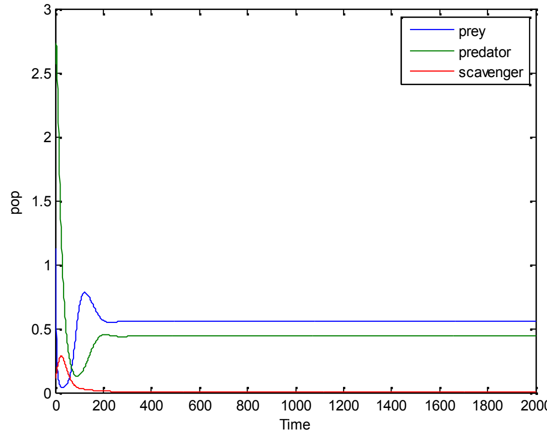


Fig2 Time series of the solution of system (2.2) approaches asymptotically to the scavenger free accumulation point $E_2 = (0.54, 0.420 , 0)$ for the datagiven in (2.5a) with $b = 0.4$.

On the other hand the maximum attack of predator rate in the range 0.6 <γ< 0.9 ,keeping other parameters as data given in (2.5a) ,causes extinction in the predator and scavenger and the system will approach to the axial accumulation point. However for 0.1 < γ< 0.6 , it is observed that system (2.2) still approach asymptotically to the positive accumulation point.

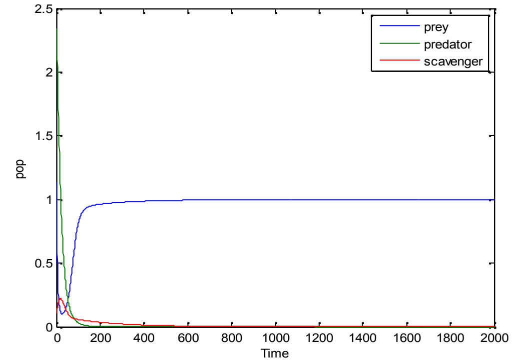


Fig3 Time series of the solution of system (2.2) approaches asymptotically to the axial accumulation point $E_1 = (0.84, 0 , 0)$ for the datagiven in (2.5a) with $\gamma = 0.4$

The death rate of predator in the range 0.5 <δ< 0.9, keeping other parameters as data given in (2.5a), causes extinction in the predator and scavenger and the system will approach to the axial accumulation point. However for 0.1 < δ< 0.5, it is observed that system (2.2) still approach asymptotically to the positive accumulation point.

The harvesting rate of predator in the range 0.7< μ < 0.9, keeping other parameters as data given in (2.5a) ,causes extinction in the predator and scavenger and the system will approach to the axial accumulation point. However for μ=0.6, it is observed that system (2.2) still approach asymptotically to the scavenger free accumulation point. Further for 0.1< μ < 0.6 the solution of the system (2.2) approaches to the positive accumulation point

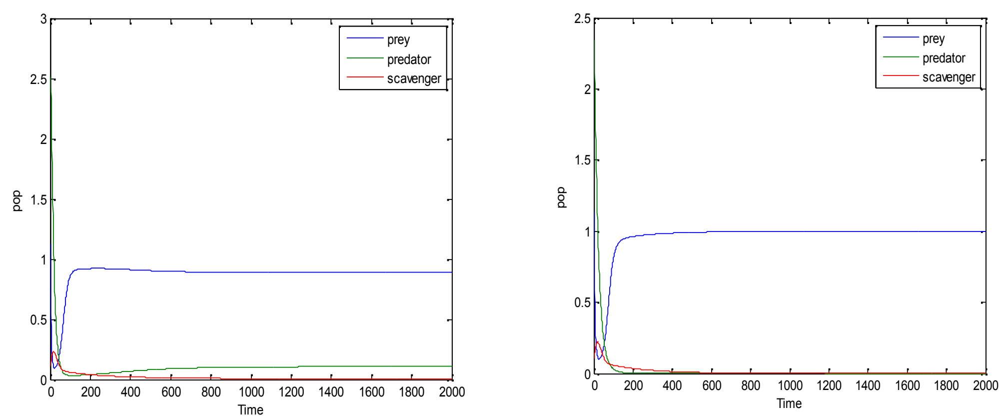


Fig4 Time series of the solution of system (2.2) approaches asymptotically to the scavenger free accumulation point $E_2 = (0.54, 0.420 , 0)$ for the datagiven in (2.5a) with $\mu = 0.6$

While when μ = 0.75 the solution of system (2.2) approaches asymptotically to the axial accumulation point $E_2 = (0.84, 0 , 0)$ for the datagiven in (2.5a)

The maximum attack rate for scavenger in the range 0.1< α< 0.4 ,keeping other parameters as data given in (2.5a) ,causes extinction in the scavenger and the system will approach to the scavenger free accumulation point .However for 0.5 < α< 0.6 , it is observed that system (2.2) still approach asymptotically to the positive accumulation point. Further for 0.6< α< 0.9 the solution of the system

(2.2) approaches to the predator free accumulation point as shown in the following figure.

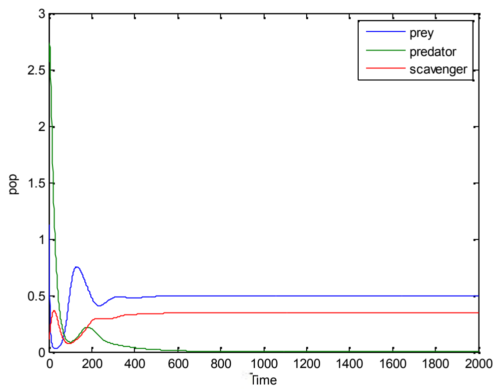


Fig5 Time series of the solution of system (2.2) approaches asymptotically to the infected free accumulation point E_4 = (0.54, 0 , 0.33) for the datagiven in (2.5a) with α = 0.4.

The harvesting rate of scavenger in the range 0.25 < E < 0.9 ,keeping other parameters as data given in (2.5a) ,causes extinction in the scavenger and the system will approach to the scavenger free accumulation point. However for 0.1<E<0., it is observed that system (2.2) still approach asymptotically to the positive accumulation point

CONCLUSIONS

In this paper, model have been discussed prey- predator and scavenger model with linear harvesting on predator and scavenger populations are discussed. the model it is found that the dynamics of the predation were dependent on the value for the basic eigenvalues, it is observed that axial accumulation state , exist always and eigenvaluves are negative and all the trajectories will be approaching towards the (DFE) E_1 if the hold the condition (2.9c). The local and global stability of the axal accumulation point is also discussed.

There is the scavenger free accumulation point E_2 of system(2.2) exist provided that the condition (2.2e) is hold . The local and global stability of the scavenger accumulation point is also discussed by using Lyapunov function. Further there is the predator free accumulation point E_3 of system(2.2) exist provided that the conditions (2.3d) ,(2.3f) is hold . The local and global stability of the scavenger accumulation point is also discussed by using Lyapunov function

While there is the positive accumulation point E_4 of system(2.2) exist provided that the conditions (2.4e) ,(2.4h) is hold . The local and global stability of the positive accumulation point is also discussed by using Lyapunov function and also study cases of all parameters. Further, the effect of maximum attack rate is also seen on the prey population. The prey population gradually decreases and predator and scavenger population increases as the predation rate increases. But as we induce harvesting in the predator and scavenger populations, the prey population suddenly decreases to a very lower level. It was noticed that the harvest works to stabilize the system, and this gives great importance to the harvest. Thus, we can conclude that harvesting rate plays a very important role for a stability to occur and this stability can be controlled by harvesting

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