I.J.S.N., VOL.8 (3) 2017: 520-535 ISSN 2229 – 6441

INTERNATIONAL JOURNAL OF SCIENCE AND NATURE

www.scienceandnature.org

THE BIFURCATION OF A FOOD CHAIN ACROSS A REFUGE STAGE-STRUCTURE PREY-PREDATOR MODEL

Azhar Abbas Majeed & Sabah Ali Rahi
Department Of Mathematics, College Of Science, University of Baghdad, Baghdad, Iraq.

ABSTRACT
In the present paper, the conditions which guarantee the occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork)of all equilibrium points of ecological mathematical model consisting of prey-predator model with two different functional responses incorporating a prey refuge are confirmed, it is observed that there is a transcritical bifurcation near the vanishing equilibrium point E_0, while there is either a transcritical or a pitchfork bifurcation near the free top-predator's and free predators' equilibrium points E_1 and E_2 respectively, on the other hand there is a saddle-node bifurcation at the coexistence equilibrium point E_3. Further investigation with special emphasis on the Hopf bifurcation near the coexistence equilibrium point is established and carried out. Finally, some numerical simulations are used to illustration the occurrence of local bifurcation of this model.

KEYWORDS: Ecological model, Sotomayor's theorem, Equilibrium point, Local bifurcation, Hopf bifurcation.

INTRODUCTION
The word bifurcation linguistically, means a kind of branching process in which the point or area forked and divided into several parts or branches, is extensively used to describe any situation or phenomena that is the qualitative, topological portrait of the topic we are discussing alters with a small change of the parameters on which the topic depends. The topics in question can be extremely various: for example, curves or surfaces, real or complex, functions or maps, vector fields, differential or integral equations. In this presentation the topic in question will be dynamic system with a formula of differential equations. Such dynamical systems widely arise in the sciences when one formulates system including equations of motion to a physical or a mathematical model. The setting of these equations is the portion or phase space of the system. In the phase space a point x corresponds to all possible states for the system, and the solution with initial condition x_0 specifies a curve in the phase space passing through x in the case of a differential equation. The universal representation of these curves which is corresponding to all points in phase space constitutes the phase portrait. This portrait gives a global qualitative image of the dynamics, and this image depends on any parameters in the equations of motion or boundary conditions. When one varies any of these parameters may result a slight deformation in the phase portrait without changing its qualitative (*i.e.*, topological) features, or the dynamics may be altered significantly, producing a qualitative deform in the phase portrait. Bifurcation theory studies these changes of the qualitative in the phase portrait, such as the appearance or disappearance of equilibriums, periodic orbits, or more complex features *e.g.*, strange attractors. Actually, the fundamentals to an understanding of nonlinear dynamical systems depend on the methods and results of bifurcation theory and this theory can likely be applied in any field of nonlinear system in nature.

Moreover, the bifurcations are divided into classes' parts: local and global bifurcation; A local bifurcation (such as saddle node, transcritical, and pitchfork) theory indicates to the bifurcations from the equilibrium points which occur in the neighborhood of a single point. This constraint overlooks a large area on global bifurcations where some qualitative changes occur in the phase portrait that are not noticed or picked up by looking near a single point. Wiggins (1988) provides an introduction to this portion of the subject. In addition, a Hopf bifurcation means the appearance or the disappearance of a periodic orbit across a local change in the properties of the stability around a fixed point. More accurately, it is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues (of the linearization around the fixed point) cross the complex plane imaginary axis [1], Under logical general propositions about the dynamical system, a small- range limits cycle branches from the fixed point. A Hopf bifurcation is also known as a Poincare–Andronov–Hopf bifurcation, named after Henri Poincaré, Eberhard Hopf, and Aleksandr Andronov.

On the other hand, the bifurcation theory is a topic with classical mathematical roots, such as in the work of Euler (1744), the characterization was presented twenty years ago by Arnold (1972), but the modern progressing of the topic introduced by Poincare with the qualitative theory of differential equations [2].

Lately, this theory has undergone a huge development by using and infusing new ideas and methods into dynamical systems theory. Naji and Majeed [3] studied the occurrence of local bifurcation near each of the equilibrium points of a prey-predator model with a refuge-stage structure in prey population. The local and Hopf bifurcation near each of the equilibrium points of a stage structured prey food web model with refuge is discussed by Kadhim, Majeed and Naji [4]. Majeed and Ali [5] discussed the local bifurcation near each of the equilibrium points and the Hopf bifurcation near the positive point of a stage structured prey food chain model with refuge and two functional responses which represent the relationship between the two predators with the non-refugees prey.

Finally, in this paper, a set of basic results and methods in local bifurcation theory around all equilibrium points and a Hopf bifurcation theory around the positive equilibrium point for system which depends on a single parameter μ is presented and discussed of a mathematical model proposed by Majeed and Rahi [6].

Mathematical Model [6]

An ecological mathematical model consisting of prey-predator model with two different functional responses incorporating a prey refuge is proposed and analyzed in [6].

$$\left.\begin{aligned}\frac{dX_1}{dT} &= r\,X_2\left(1-\frac{X_2}{k}\right)-\frac{a_1\,(1-m)\,X_1}{b+X_1}\,Y_1 - s\,X_1 - d_1\,X_1\\ \frac{dX_2}{dT} &= s\,X_1 - a_2\,(1-m)\,X_2\,Y_1 - d_2\,X_2\\ \frac{dY_1}{dT} &= \frac{e_1 a_1\,(1-m)\,X_1}{b+X_1}\,Y_1 + e_2 a_2\,(1-m)\,X_2\,Y_1 - a_3\,Y_1\,Y_2 - d_3\,Y_1\\ \frac{dY_2}{dT} &= e_3 a_3\,Y_1\,Y_2 - d_4\,Y_2\,.\end{aligned}\right\}\ (1)$$

With initial conditions $X_i\,(0) \geq 0$ and $Y_i(0) \geq 0\,,\ i = 1\,,2\,.$

Note that the above proposed model has fourteen parameters in all which make the analysis difficult. So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$$t = r\,\mathrm{T}\ ,\ u_1 = \frac{\mathrm{b}}{\mathrm{k}}\ ,\ u_2 = \frac{\mathrm{s}}{\mathrm{r}}\ ,\ u_3 = \frac{\mathrm{d}_1}{\mathrm{r}}\ ,\ u_4 = \frac{\mathrm{a}_2\,\mathrm{k}}{\mathrm{a}_1}\ , u_5 = \frac{\mathrm{d}_2}{\mathrm{r}}\ ,\ u_6 = \frac{e_1\,a_1}{r}\ ,$$
$$u_7 = \frac{e_2 a_2 k}{r}\ u_8 = \frac{d_3}{r}\ ,\ u_9 = \frac{e_3 a_3 k}{r}\ ,\ u_{10} = \frac{d_4}{r}\ , x = \frac{X_1}{k}\ ,$$
$$y = \frac{X_2}{k}\ , z = \frac{a_1\,Y_1}{rk}\ , w = \frac{a_3\,Y_2}{r}.$$

Then the non-dimensional form of system (1) can be written as:

$$\left.\begin{aligned}\frac{dx}{dt} &= x\left[\frac{y\,(1-y)}{x}-\frac{(1-m)z}{u_1+x}-(u_2+u_3)\right] = f_1(x,y,z,w)\\ \frac{dy}{dt} &= y\left[\frac{u_2\,x}{y}-u_4(1-m)\,z-u_5\right] = f_2(x,y,z,w)\\ \frac{dz}{dt} &= z\left[\frac{u_6\,(1-m)x}{u_1+x}+u_7(1-m)y-w-u_8\right] = f_3(x,y,z,w)\\ \frac{dw}{dt} &= w\,[\,u_9 z - u_{10}\,] = f_4(x,y,z,w).\end{aligned}\right\}\quad (2)$$

With $x(0) \geq 0\,, y(0) \geq 0\,, z(0) \geq 0$ and $w(0) \geq 0\,.$

It is observed that the number of parameters has been reduced from fourteen in the system (1) to eleven in the system (2).

Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional spaces.

$R^4_+ = \{(x,y,z,w) \in R^4 : x(0) \geq 0, y(0) \geq 0\,, z(0) \geq 0\,, w(0) \geq 0\}$.

Therefore these functions are Lipschitzian on R^4_+, and hence the solution of the system (2) exists and is unique. Further, all the solutions of system (2) with non-negative initial conditions are uniformly bounded as shown in theorem (1) which is proved in [6].

The local bifurcation analysis of system (2)

In this section, the influence of altering the parameter values on the dynamical behavior of the system (2) around each equilibrium point is discussed. Recall that the existence of non-hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application of the Sotomayor's theorem [7] for local bifurcation is appropriate.

Now, according to Jacobian matrix of system (2) given in eq. (6) (more details see [6]), it is clear to verify that for any nonzero vector $V=(v_1,v_2,v_3,v_4)^T$ we have:

$$D^2 f_\mu(X,\mu)(V,V)=[e_{ij}]_{4\times 1} \quad , \qquad (3.1)$$

where:

$$e_{11}=2\left[\frac{u_1(1-m)v_1}{(u_1+x)^2}\left(\frac{zv_1}{u_1+x}-v_3\right)-v_2^2\right],$$
$$e_{21}=-2\,u_4(1-m)v_2v_3,$$
$$e_{31}=2\left[\frac{u_1u_6(1-m)\,v_1}{(u_1+x)^2}\left(v_3-\frac{zv_1}{u_1+x}\right)+v_3(u_7(1-m)v_2-v_4)\right],$$
$$e_{41}=2\,u_9v_3v_4\,,$$

and

$$D^3 f_\mu(X,\mu)(V,V,V)=\begin{pmatrix} 6\frac{u_1(1-m)v_1^2}{(u_1+x)^3}\left[v_3-\frac{z\,v_1}{u_1+x}\right] \\ 0 \\ 6\frac{u_1u_6(1-m)v_1^2}{(u_1+x)^3}\left[\frac{z\,v_1}{u_1+x}-v_3\right] \\ 0 \end{pmatrix}. \qquad (3.2)$$

Where $X=(x,y,z,w)^T$ and μ is any bifurcation parameter.

In the following theorems the local bifurcation conditions near the equilibrium points are established.

Theorem (3.1): If the parameter u_5 passes through the value $u_5^\circ=\frac{u_2}{u_2+u_3}$ then the vanishing equilibrium point E_0 transforms into non-hyperbolic equilibrium point and system (2) possesses a transcritical bifurcation but neither saddle-node, nor pitchfork bifurcation can occur at E_0.

Proof: According to the Jacobian matrix $J(E_0)$ given by eq.(7 a) given in [6] the system(2) at the equilibrium point E_0 has zero Eigenvalue (say $\lambda_{0y}=0$) at $u_5=u_5^\circ$, and the Jacobian matrix J_0 with $u_5=u_5^\circ$ becomes:

$$J_0^\circ=J(u_5=u_5^\circ)=\begin{pmatrix} -(u_2+u_3) & 1 & 0 & 0 \\ u_2 & -u_5^\circ & 0 & 0 \\ 0 & 0 & -u_8 & 0 \\ 0 & 0 & 0 & -u_{10} \end{pmatrix}.$$

Now, let $V^{[0]}=\left(v_1^{[0]},v_2^{[0]},v_3^{[0]},v_4^{[0]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{0y}=0$. Thus $\left(J_0^\circ-\lambda_{0y}I\right)V^{[0]}=0$, which gives:

$v_2^{[0]}=\frac{u_2}{u_5^\circ}v_1^{[0]}$, $v_3^{[0]}=0$, $v_4^{[0]}=0$ and $v_1^{[0]}$ any nonzero real number.

Let $\Psi^{[0]}=\left(\psi_1^{[0]},\psi_2^{[0]},\psi_3^{[0]},\psi_4^{[0]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{0y}=0$ of the matrix $J_0^{\circ T}$. Then we have, $(J_0^{\circ T}-\lambda_{0y}I)\Psi^{[0]}=0$. By solving this equation for $\Psi^{[0]}$ we obtain, $\Psi^{[0]}=\left(\psi_1^{[0]},\frac{1}{u_5^\circ}\psi_1^{[0]},0,0\right)^T$, where $\psi_1^{[0]}$ any nonzero real number.

Now, consider:

$$\frac{\partial f}{\partial u_5}=f_{u_5}(X,u_5)=\left(\frac{\partial f_1}{\partial u_5},\frac{\partial f_2}{\partial u_5},\frac{\partial f_3}{\partial u_5},\frac{\partial f_4}{\partial u_5}\right)^T=(0,-y,0,0)^T .$$

So, $f_{u_5}(E_0,u_5^\circ)=(0,0,0,0)^T$ and hence $\left(\Psi^{[0]}\right)^T f_{u_5}(E_0,u_5^\circ)=0$.

Therefore, according to Sotomayor's theorem the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied.

Now, since

$$Df_{u_5}(X\,,u_5)=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\,,$$

where $Df_{u_5}(X\,,u_5)$ represents the derivative of $f_{u_5}(X,u_5)$ with respect to $X=(x\,,y\,,z\,,w)^T$. Further, it is observed that

$$Df_{u_5}(\,E_0\,,u_5^{\circ})V^{[0]}=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}v_1^{[0]}\\ \frac{u_2}{u_5^{\circ}}v_1^{[0]}\\0\\0\end{pmatrix}=\begin{pmatrix}0\\-\frac{u_2}{u_5^{\circ}}v_1^{[0]}\\0\\0\end{pmatrix}\,,$$

$$\left(\Psi^{[0]}\right)^T\left[Df_{u_5}(\,E_0\,,u_5^{\circ})V^{[0]}\right]=-\frac{u_2}{(u_5^{\circ})^2}\,v_1^{[0]}\psi_1^{[0]}\neq 0\ .$$

Now, by substituting $V^{[0]}$ in (3.1) we get:

$$D^2f(\,E_0\,,u_5^{\circ})(V^{[0]}\,,V^{[0]})=\begin{pmatrix}-2\left(\frac{u_2}{u_5^{\circ}}\right)^2\left(v_1^{[0]}\right)^2\\0\\0\\0\end{pmatrix}.$$

Hence, it is obtained that:

$$\left(\Psi^{[0]}\right)^TD^2f(\,E_0\,,u_5^{\circ})(V^{[0]}\,,V^{[0]})=-2\left(\frac{u_2}{u_5^{\circ}}\right)^2\left(v_1^{[0]}\right)^2\psi_1^{[0]}\neq 0\ .$$

Thus, according to Sotomayor's theorem system $(\,2\,)$ has transcrirtical bifurcation but not experience a pitchfork bifurcation at E_0 with the parameter $u_5=u_5^{\circ}$.

Theorem (3.2): Suppose that the following conditions are satisfied:

$$\frac{u_4(1-m)\bar{y}}{u_5}\neq S_2 \qquad (\,3\,.2\,a\,)$$

$$S_4\neq S_5 \qquad (\,3\,.2\,b\,)$$

$$\psi_1^{[1]}\neq u_6\Psi_3^{[1]} \qquad (\,3\,.2\,c\,)$$

where:

$$S_1=\frac{u_4(1-m)\bar{y}-u_5\,S_2}{u_2},S_2=\frac{(1-m)[u_4(u_2\ +u_3)(u_1+\bar{x}\,)\bar{y}+u_2\bar{x}]}{u_2(u_1+\bar{x}\,)\bar{y}},$$

$$S_3=\frac{u_2\ +u_3}{u_2}\,,\qquad S_4=\left[u_4\,(1-m)S_2S_3\,\psi_1^{[1]}-\frac{u_1u_6(1-m)S_1}{(u_1+\bar{x})^2}\,\psi_3^{[1]}\right].$$

$$S_5=\left[\left(\frac{u_1(1-m)S_1}{(u_1+\bar{x})^2}+{S_2}^2\right)\psi_1^{[1]}+u_7(1-m)S_2\,\psi_3^{[1]}\right].$$

Then system (2) at the equilibrium point $E_1=(\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ with the parameter $\bar{u}_8=\frac{u_6(1-m)\bar{x}}{(u_1+\bar{x})}+u_7(1-m)\bar{y}$ possesses a transcritical or a pitchfork bifurcation but no saddle-node bifurcation can occur at $E_1=(\,\bar{x}\,,\bar{y}\,,0\,,0\,)$.

Proof: According to the Jacobian matrix J_1 given by eq.$(\,2.8\,a\,)$ the system $(\,2\,)$ at the equilibrium point $E_1=(\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ has zero eigenvalue (say $\lambda_{1z}=0$) at $u_8=\bar{u}_8$, and the Jacobian matrix J_1 with $u_8=\bar{u}_8$ becomes:

$$\bar{J}_1=J_1(\,u_8=\bar{u}_8)=\begin{bmatrix}-(u_2+u_3)&1-2\bar{y}&\frac{-(1-m)\bar{x}}{(u_1+\bar{x})}&0\\u_2&-u_5&-u_4(1-m)\bar{y}&0\\0&0&0&0\\0&0&0&-u_{10}\end{bmatrix}.$$

Let $V^{[1]}=\left(v_1^{[1]}\,,v_2^{[1]}\,,v_3^{[1]}\,,v_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1z}=0$. Thus $(\bar{J}_1-\lambda_{1z}I)V^{[1]}=0$, which gives:

$v_1^{[1]}=S_1v_3^{[1]}$, $v_2^{[1]}=-S_2v_3^{[1]}$ and $v_4^{[1]}=0$, where $v_3^{[1]}$ any nonzero real number.

Clearly, $S_2>0$, while $S_1>0$ if the condition $(\,3\,.2\,a\,)$ is satisfied, where S_1 and S_2 which are mentioned in the state of the theorem.

Let $\Psi^{[1]}=\left(\psi_1^{[1]}\,,\psi_2^{[1]}\,,\psi_3^{[1]}\,,\psi_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1Z}=0$ of the matrix ${\bar{J}_1}^T$.

Then we have $\left(\bar{J}_1^{\,T}-\lambda_{1Z}I\right)\Psi^{[1]}=0$. By solving this equation for $\Psi^{[1]}$ we obtain $\Psi^{[1]}=\left(\psi_1^{[1]},S_3\,\psi_1^{[1]},\psi_3^{[1]},0\right)^T$, where $\psi_1^{[1]}$ *and* $\psi_3^{[1]}$ are any nonzero real numbers, with S_3 which is mentioned in the state of the theorem.
Now, consider:

$$\frac{\partial f}{\partial u_8}=f_{u_8}(X,u_8)=\left(\frac{\partial f_1}{\partial u_8},\frac{\partial f_2}{\partial u_8},\frac{\partial f_3}{\partial u_8},\frac{\partial f_4}{\partial u_8}\right)^T=(0,0,-z,0)^T\ .$$

So, $f_{u_8}(E_1,\bar{u}_8)=(0,0,0,0)^T$ and hence $\left(\Psi^{[1]}\right)^T f_{u_8}(E_1,\bar{u}_8)=0$.

Therefore, according to Sotomayor's theorem the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{u_8}(X,u_8)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix},$$

where $Df_{u_8}(X,u_8)$ Represents the derivative of $f_{u_8}(X,u_8)$ with respect to $X=(x,y,z,w)^T$.
Further, it is observed that

$$Df_{u_8}(E_1,\bar{u}_8)V^{[1]}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}S_1v_3^{[1]}\\-S_2v_3^{[1]}\\v_3^{[1]}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\-v_3^{[1]}\\0\end{pmatrix},$$

$$\left(\Psi^{[1]}\right)^T\left[Df_{u_8}(E_1,\bar{u}_8)V^{[1]}\right]=-v_3^{[1]}\psi_3^{[1]}\neq 0\ .$$

Now, by substituting $V^{[1]}$ in (3.1) we get:

$$D^2f(E_1,\bar{u}_8)\left(V^{[1]},V^{[1]}\right)=\begin{pmatrix}-2\left[\frac{u_1(1-m)S_1}{(u_1+\bar{x})^2}+S_2^{\ 2}\right]\left(v_3^{[1]}\right)^2\\2u_4\,(1-m)S_2\left(v_3^{[1]}\right)^2\\2\left[\frac{u_1u_6(1-m)S_1}{(u_1+\bar{x})^2}-u_7(1-m)S_2\right]\left(v_3^{[1]}\right)^2\\0\end{pmatrix}.$$

Hence, it is obtained that:

$$\left(\Psi^{[1]}\right)^T D^2f(E_1,\bar{u}_8)\left(V^{[1]},V^{[1]}\right)=2[S_4-S_5]\left(v_3^{[1]}\right)^2,$$

where S_4 and S_5 are mentioned in the state of the theorem.
So, if in addition to the condition $(3.2\,a)$, the condition $(3.2\,b)$ is satisfied we obtain that:
$\left(\Psi^{[1]}\right)^T D^2f(E_1,\bar{u}_8)\left(V^{[1]},V^{[1]}\right)\neq 0$.
Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at the equilibrium point $E_1=(\bar{x},\bar{y},0,0)$ with the parameter:

$$\bar{u}_8=\frac{u_6(1-m)\bar{x}}{(u_1+\bar{x})}+u_7(1-m)\bar{y}\,.$$

Now, by reserving the condition $(3.2\,b)$ and substituting $V^{[1]}$ in (3.2) we get:

$$D^3f\left(V^{[1]},V^{[1]},V^{[1]}\right)=\begin{pmatrix}\frac{6u_1(1-m)S_1^2}{(u_1+\bar{x})^3}\left(v_3^{[1]}\right)^3\\0\\\frac{-6\,u_1u_6(1-m)S_1^2}{(u_1+\bar{x})^3}\left(v_3^{[1]}\right)^3\\0\end{pmatrix}.$$

So,

$$\left(\Psi^{[1]}\right)^T D^3f(E_1,\bar{u}_8)\left(V^{[1]},V^{[1]},V^{[1]}\right)=\frac{6u_1(1-m)S_1^2}{(u_1+\bar{x})^3}\left[\psi_1^{[1]}-u_6\,\psi_3^{[1]}\right]\left(v_3^{[1]}\right)^3.$$

So, if the condition $(3.2\,c)$ is satisfied we obtain that:

$$\left(\Psi^{[1]}\right)^T D^3f(E_1,\bar{u}_8)\left(V^{[1]},V^{[1]},V^{[1]}\right)\neq 0\,.$$

Thus, according to Sotomayor's theorem system (2) has a pitchfork bifurcation at the equilibrium point $E_1=(\bar{x},\bar{y},0,0)$ with the parameter:

$$\bar{u}_8 = \frac{u_6(1-m)\bar{x}}{(u_1+\bar{x})} + u_7(1-m)\bar{y}\,.$$

Theorem (3.3): Suppose that the following conditions are satisfied:

$$-\frac{c_{11}\hat{x}}{c_{12}} < \hat{y} < \frac{1}{2} \qquad (3.3\,a)$$

$$K_1 \neq \frac{K_2}{R_4} \qquad (3.3\,b)$$

$$\hat{z} \neq \frac{(u_1+\hat{x})R_3}{R_1} \qquad (3.3\,c)$$

where:

$$R_1 = \hat{z}[c_{22}c_{13} - c_{23}c_{12}]\ ,\ R_2 = \hat{z}[c_{11}c_{23} - c_{13}c_{21}],$$
$$R_3 = \hat{z}[c_{12}c_{21} - c_{11}c_{22}]\ ,\ R_4 = c_{31}[c_{22}c_{13} - c_{23}c_{12}] + c_{32}[c_{11}c_{23} - c_{13}c_{21}]\,,$$

with,

$$K_1 = \frac{u_1(1-m)R_1^2\,\hat{z}}{(u_1+\hat{x})^3R_4}\psi_1^{[2]} + u_9R_3\psi_4^{[2]},$$
$$K_2 = \left[\frac{u_1(1-m)R_1R_3}{(u_1+\hat{x})^2} + R_2^2 - \frac{c_{11}u_4(1-m)R_2R_3}{u_2}\right]\psi_1^{[2]}.$$

Then system (2) at the free top-predator's equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ with the parameter $\hat{u}_{10} = u_9\hat{z}$ possesses a transcritical or a pitchfork bifurcation but no saddle-node bifurcation can occur at $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$.

Proof: According to the Jacobian matrix J_2 given by eq.($2.9\,a$) the system (2) at the equilibrium point E_2 has zero eigenvalue (say $\lambda_{2w} = 0$) at $u_{10} = \hat{u}_{10}$, and the Jacobian matrix J_2 with $u_{10} = \hat{u}_{10}$ becomes:

$$\hat{J}_2 = J_2(u_{10} = \hat{u}_{10}) = [\hat{c}_{ij}]_{4\times 4}\,,$$

where $\hat{c}_{ij} = c_{ij}$ for all $i, j = 1, 2, 3, 4$ except $\hat{c}_{44} = 0$.

Let $V^{[2]} = \left(v_1^{[2]}, v_2^{[2]}, v_3^{[2]}, v_4^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2w} = 0$. Thus $\left(\hat{J}_2 - \lambda_{2w}I\right)V^{[2]} = 0$, which gives:

$v_1^{[2]} = \frac{R_1}{R_4}v_4^{[2]}$, $v_2^{[2]} = \frac{R_2}{R_4}v_4^{[2]}$, $v_3^{[2]} = \frac{R_3}{R_4}v_4^{[2]}$, where $v_4^{[2]}$ is any nonzero real number, with R_i; $i = 1, 2, 3, 4$ which are mentioned in the state of the theorem

Clearly, $R_2 > 0$, while $R_i > 0$; $i = 1, 3, 4$ if the condition ($3.3\,a$) is satisfied.

Let $\Psi^{[2]} = \left(\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{2w} = 0$ of the matrix $\hat{J}_2^{\,T}$. Then we have $\left(\hat{J}_2^{\,T} - \lambda_{2w}I\right)\Psi^{[2]} = 0$. By solving this equation for $\Psi^{[2]}$ we obtain, $\Psi^{[2]} = \left(\psi_1^{[2]}, \frac{-c_{11}}{u_2}\psi_1^{[2]}, 0, \psi_4^{[2]}\right)^T$, where $\psi_1^{[2]}$ and $\psi_4^{[2]}$ are any nonzero real numbers.

Now, consider:

$$\frac{\partial f}{\partial u_{10}} = f_{u_{10}}(X, u_{10}) = \left(\frac{\partial f_1}{\partial u_{10}}, \frac{\partial f_2}{\partial u_{10}}, \frac{\partial f_3}{\partial u_{10}}, \frac{\partial f_4}{\partial u_{10}}\right)^T = (0, 0, 0, -w)^T\,.$$

So, $f_{u_{10}}(E_2, \hat{u}_{10}) = (0, 0, 0, 0)^T$ and hence $\left(\Psi^{[2]}\right)^T f_{u_{10}}(E_2, \hat{u}_{10}) = 0$.

Therefore, according to Sotomayor's theorem the saddle-node bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{u_{10}}(X, u_{10}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where $Df_{u_{10}}(X, u_{10})$ represents the derivative of $f_{u_{10}}(X, u_{10})$ with respect to $X = (x, y, z, w)^T$.

Further, it is observed that

$$Df_{u_{10}}(E_2, \hat{u}_{10})V^{[2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{R_1}{R_4} v_4^{[2]} \\ \frac{R_2}{R_4} v_4^{[2]} \\ \frac{R_3}{R_4} v_4^{[2]} \\ v_4^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -v_4^{[2]} \end{pmatrix},$$

$$\left(\Psi^{[2]}\right)^T \left[Df_{u_{10}}(E_2, \hat{u}_{10})V^{[2]}\right] = -v_4^{[2]} \psi_4^{[2]} \neq 0 .$$

Now, by substituting $V^{[2]}$ in (3.1) we get:

$$D^2 f(E_2, \hat{u}_{10})\left(V^{[2]}, V^{[2]}\right) = \left[\square_{ij}\right]_{4\times 1},$$

where:

$$\square_{11} = \frac{2}{R_4^2}\left[\frac{u_1(1-m)R_1}{(u_1+\hat{x})^2}\left(\frac{R_1\hat{z}}{(u_1+\hat{x})} - R_3\right) - R_2^2\right]\left(v_4^{[2]}\right)^2$$

$$\square_{21} = -\frac{2}{R_4^2}\left[u_4(1-m)R_2R_3\right]\left(v_4^{[2]}\right)^2$$

$$\square_{31} = \frac{2}{R_4}\left[\frac{u_1u_6(1-m)R_1}{(u_1+\hat{x})^2 R_4}\left(R_3 - \frac{R_1\hat{z}}{(u_1+\hat{x})}\right) + R_3\left(\frac{u_7(1-m)R_2}{R_4} - 1\right)\right]\left(v_4^{[2]}\right)^2$$

$$\square_{41} = \frac{2}{R_4}\left(u_9R_3\right)\left(v_4^{[2]}\right)^2,$$

Hence, it is obtained that:

$$\left(\Psi^{[2]}\right)^T D^2 f(E_2, \hat{u}_{10})\left(V^{[2]}, V^{[2]}\right) = \frac{2}{R_4}\left(K_1 - \frac{K_2}{R_4}\right)\left(v_4^{[2]}\right)^2.$$

with, K_2 and K_3 are mentioned in the state of the theorem.

So, according to the condition ($3.3\,b$) we obtain that:

$$\left(\Psi^{[2]}\right)^T D^2 f(E_2, \hat{u}_{10})\left(V^{[2]}, V^{[2]}\right) \neq 0 .$$

Thus, by using Sotomayor's theorem system (2) has transcrirtical bifurcation at the free top-predator's equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ with the parameter $\hat{u}_{10} = u_9\hat{z}$.

Now, by reserving the condition ($3.3\,b$) and substituting $V^{[2]}$ in (3.2) we get:

$$D^3 f(E_2, \hat{u}_{10})\left(V^{[2]}, V^{[2]}, V^{[2]}\right) = \left[l_{ij}\right]_{4\times 1},$$

where:

$$l_{11} = \frac{6u_1(1-m)R_1^2}{(u_1+\hat{x})^3R_4^3}\left[R_3 - \frac{R_1\hat{z}}{(u_1+\hat{x})}\right]\left(v_4^{[2]}\right)^3, \qquad l_{21} = 0,$$

$$l_{31} = \frac{6u_1u_6(1-m)R_1^2}{(u_1+\hat{x})^3R_4^3}\left[\frac{R_1\hat{z}}{(u_1+\hat{x})} - R_3\right]\left(v_4^{[2]}\right)^3, \quad l_{41} = 0,$$

So,

$$\left(\Psi^{[2]}\right)^T D^3 f\left(V^{[2]}, V^{[2]}, V^{[2]}\right) = l_{11}\,\psi_1^{[2]}.$$

So, according to the condition ($3.3\,c$) we obtain that:

$$\left(\Psi^{[2]}\right)^T D^3 f\left(V^{[2]}, V^{[2]}, V^{[2]}\right) \neq 0 .$$

Thus, by using Sotomayor's theorem system (2) has a pitchfork bifurcation at the free top-predator's equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ with the parameter $\hat{u}_{10} = u_9\hat{z}$.

Theorem (3.4): Suppose that the following conditions are satisfied:

$$y^* < \frac{1}{2}, \qquad (3.4\,a)$$

$$u_2 > \frac{u_4(1-m)F\,z^*}{(1-2\,y^*)\,(u_1+x^*)^2}, \qquad (3.4\,b)$$

$$\frac{u_1(1-m)L_1^2\,z^*}{(u_1+x^*)^3} \neq 1, \qquad (3.4\,c)$$

where:

$$F = u_1(1-m)\,z^* + (u_2+u_3)\,(u_1+x^*)^2,$$

$$L_1 = \frac{(1-2\,y^*)\,(u_1+x^*)^2}{F}, \quad L_2 = \frac{(1-m)[u_1u_6(1-2\,y^*) + u_7F]}{F}$$

$$L_3 = \frac{u_2\,(u_1+x^*)^2}{F}, \qquad L_4 = \frac{(1-m)[u_2(u_1+x^*)\,x^* + u_4y^*F]}{u_9\,w^*F}.$$

Then system (2.2) at the equilibrium point $E_3 = (x^*, y^*, z^*, w^*)$ with the parameter value: $u_5^* = \dfrac{u_2(1-2\,y^*)\,(u_1 + x^*)^2 - \mathrm{u}_4(1-\mathrm{m})\,\mathrm{z}^*\mathrm{F}}{\mathrm{F}}$, has a saddle − node bifurcation, but neither transcritical nor pitchfork bifurcation can occur at E_3.

Proof: The characteristic equation given by eq.($2.10\,b$) having zero eigenvalue (say $\lambda_2 = 0$) if and only if $B_4 = 0$ and then E_3 becomes a non-hyperbolic equilibrium point. Clearly the Jacobian matrix of system (2.2) at the equilibrium point E_3 with parameter $u_5 = u_5^*$ becomes: $J_3^* = J(\,u_5 = u_5^*) = \left[\, d_{ij}^* \,\right]_{4\times 4}$ where,
$d_{ij}^* = d_{ij}$ for all $i\,,j = 1, 2, 3, 4$ except d_{ij} which is given by:
$d_{22}^* = -u_5^* - u_4(1-m)\,z^*$.
Note that, $u_5^* > 0$ provided that conditions (3.4 a) and (3.4 b) hold.
Let $V^{[3]} = \left(v_1^{[3]}, v_2^{[3]}, v_3^{[3]}, v_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 0$. Thus $(J_3^* - \lambda_2 I)\,V^{[3]} = 0$, which gives:
$V^{[3]} = \left(L_1 v_2^{[3]}, v_2^{[3]}, 0\,, L_2 v_2^{[3]}\right)^T$, where $v_2^{[3]}$ any nonzero real number, with L_1 and L_2 which are mentioned in the state of the theorem.
Clearly, L_1 and L_2 are positive under the condition ($3.4\,a$).
Let $\Psi^{[3]} = \left(\psi_1^{[3]}, \psi_2^{[3]}, \psi_3^{[3]}, \psi_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_2 = 0$ of the matrix J_3^{*T}. Then we have $(J_3^{*T} - \lambda_2 I)\Psi^3 = 0$. By solving this equation for $\Psi^{[3]}$ we obtain:
$\Psi^{[3]} = \left(L_3\psi_2^{[3]}, \psi_2^{[3]}, 0\,, L_4\psi_2^{[3]}\right)^T$ where $\psi_2^{[4]}$ any nonzero real number,
with L_3 and L_4 which are mentioned in the state of the theorem.
Now,

$$\frac{\partial f}{\partial u_5} = f_{u_5}(X\,,u_5) = \left(\frac{\partial f_1}{\partial u_5}, \frac{\partial f_2}{\partial u_5}, \frac{\partial f_3}{\partial u_5}, \frac{\partial f_4}{\partial u_5}\right)^T = (0\,,-y\,,0\,,0)^T\ .$$

So, $f_{u_5}(E_3\,,u_5^*) = (0\,,-y^*\,,0\,,0)^T$,
and hence $\left(\Psi^{[3]}\right)^T f_{u_5}(E_3\,,u_5^*) = -y^*\psi_2^{[3]} \neq 0$.
Therefore, according to Sotomayor's theorem neither a transcritical nor a pitchfork bifurcation can occur at E_3 , while the first condition of a saddle-node bifurcation is satisfied.
Moreover, by substituting $V^{[3]}$ in (3.1) we get:

$$D^2 f(E_3\,,u_5^*)\left(V^{[3]}, V^{[3]}\right) = \begin{pmatrix} 2\left[\dfrac{u_1(\,1-m)L_1^2\,z^*}{(u_1 + x^*\,)^3} - 1\right]\left(v_2^{[3]}\right)^2 \\ 0 \\ -2\left[\dfrac{u_1 u_6(\,1-m)L_1^2\,z^*}{(u_1 + x^*\,)^3}\right]\left(v_2^{[3]}\right)^2 \\ 0 \end{pmatrix}.$$

Hence, it is obtained that:

$$\left(\Psi^{[3]}\right)^T D^2 f(E_3\,,u_5^*)\left(V^{[3]}, V^{[3]}\right) = 2\left[\frac{u_1(\,1-m)L_1^2\,z^*}{(u_1 + x^*\,)^3} - 1\right] L_3\left(v_2^{[3]}\right)^2 \psi_2^{[3]}.$$

So, according to the condition ($3\,.\,4\,c$) we obtain that:

$$\left(\Psi^{[3]}\right)^T D^2 f(E_3\,,u_5^*)\left(V^{[3]}, V^{[3]}\right) \neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has a saddle-node bifurcation at $E_3 = (\,x^*\,,\,y^*\,,\,z^*\,,\,w^*)$ with the parameter:

$$u_5^* = \frac{u_2(1-2\,y^*)\,(u_1 + x^*)^2 - \mathrm{u}_4(1-\mathrm{m})\,\mathrm{z}^*\mathrm{F}}{\mathrm{F}}.$$

The Hopf bifurcation analysis of system (2)

The occurrence of a Hopf bifurcation around the coexistence (positive) equilibrium point E_3 of system (2) is discussed in this section.

Firstly, we need to know that the Hopf bifurcation for $n = 4$ is structured according to the Haque and Venturino method [8] in order to investigate the occurrence of the Hopf bifurcation.
Consider the characteristic equation (10 b) which is given in [6]:

$$P_4(\lambda) = \lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0,$$

here $B_1 = -tr(J(x^*))$, $B_2 = M_1(J(x^*))$, $B_3 = -M_2(J(x^*))$ and $B_4 = det\,(J(x^*))$ with $M_1(J(x^*))$ and $M_2(J(x^*))$ represent the sum of the principal minors of order two and three of $J(x^*)$ respectively.
Clearly, the first condition of Hopf bifurcation satisfies if and only if:

$B_i > 0\ ;\ i = 1,3,4\ ,\ \Delta_1 = B_1B_2 - B_3 > 0\ , B_1^3 - 4\,\Delta_1 > 0$

and $\Delta_2 = B_3(B_1B_2 - B_3) - B_1^2B_4 = 0$. Consequently, $B_4 = \frac{B_3(B_1B_2-B_3)}{B_1^2}$.

So, the characteristic equation becomes:

$$P_4(\lambda) = \left(\lambda^2 + \frac{B_3}{B_1}\right)\left(\lambda^2 + B_1\lambda + \frac{\Delta_1}{B_1}\right) = 0 \qquad (4.1)$$

Clearly, the roots of eq.(4.1) are:

$$\lambda_{1,2} = \pm i\sqrt{\frac{B_3}{B_1}} \quad \text{and} \quad \lambda_{3,4} = \frac{1}{2}\left(-B_1 \pm \sqrt{B_1^2 - 4\frac{\Delta_1}{B_1}}\right).$$

Now, in order to verify the transversality condition of Hopf bifurcation, we substitute $\lambda(\mu) = \varepsilon_1(\mu) \mp i\varepsilon_2(\mu)$ into eq. (4.1), and then calculating its derivative with respect to the bifurcation parameter μ, $P_4'(\lambda(\mu)) = 0$, comparing the two sides of this equation and then equating their real and imaginary parts, we have:

$$\left.\begin{aligned} &\Psi^*(\mu)\varepsilon_1'(\mu) - \Phi^*(\mu)\varepsilon_2' + \Theta^*(\mu) = 0 \\ &\Phi^*(\mu)\varepsilon_1'(\mu) + \Psi^*(\mu)\varepsilon_2'(\mu) + \Gamma^*(\mu) = 0 \end{aligned}\right\}. \qquad (4.2)$$

Where:

$$\left.\begin{aligned} \Psi^*(\mu) &= 4(\varepsilon_1(\mu))^3 + 3B_1(\mu)(\varepsilon_1(\mu))^2 + B_3(\mu) + 2B_2(\mu)\varepsilon_1(\mu) \\ &\quad -12\varepsilon_1(\mu)(\varepsilon_2(\mu))^2 - 3B_1(\mu)(\varepsilon_2(\mu))^2 \\ \Phi^*(\mu) &= 12(\varepsilon_1(\mu))^2\varepsilon_2(\mu) + 6B_1(\mu)\varepsilon_1(\mu)\varepsilon_2(\mu) + 2B_2(\mu)\varepsilon_2(\mu) \\ &\quad -4(\varepsilon_2(\mu))^3 \\ \Theta^*(\mu) &= (\varepsilon_1(\mu))^3\, B_1'(\mu) + B_3'(\mu)\varepsilon_1(\mu) + B_2'(\mu)(\varepsilon_1(\mu))^2 \\ &\quad +B_4'(\mu) - 3B_1'(\mu)\varepsilon_1(\mu)(\varepsilon_2(\mu))^2 - B_2'(\mu)(\varepsilon_2(\mu))^2 \\ \Gamma^*(\mu) &= 3B_1'(\mu)(\varepsilon_1(\mu))^2\varepsilon_2(\mu) + B_3'(\mu)\varepsilon_2(\mu) + 2B_2'(\mu)\varepsilon_1(\mu)\varepsilon_2(\mu) \\ &\quad -B_1'(\mu)(\varepsilon_2(\mu))^3 \end{aligned}\right\}. \qquad (4.3)$$

Solving the linear system (4.2) by using Cramer's rule for the unknowns $\varepsilon_1'(\mu)$ and $\varepsilon_2'(\mu)$, gives that:

$$\varepsilon_1'(\mu) = \frac{\Theta^*(\mu)\,\Psi^*(\mu) + \Gamma^*(\mu)\,\Phi^*(\mu)}{(\Psi^*(\mu))^2 + (\Phi^*(\mu))^2} \quad \text{and} \quad \varepsilon_2'(\mu) = \frac{-\Gamma^*(\mu)\,\Psi^*(\mu) + \Theta^*(\mu)\,\Phi^*(\mu)}{(\Psi^*(\mu))^2 + (\Phi^*(\mu))^2}.$$

Hence, the second condition of the Hopf bifurcation which is necessary and sufficient condition (transversality condition) $\frac{d}{d\mu}Re(\lambda)\,\big|_{\mu=\bar{\mu}} = \varepsilon_1'(\mu)\,\big|_{\mu=\bar{\mu}}$ not being zero if and only if:

$$\Theta^*(\mu)\,\Psi^*(\mu) + \Gamma^*(\mu)\,\Phi^*(\mu) \neq 0\ . \qquad (4.4)$$

Moreover, according to the above results the occurrence of Hopf bifurcation near the positive equilibrium point is carried out as shown in the following theorem.

Theorem (4.1): Suppose that the locally conditions (2.10 c), (2.10 d) and(2.10 e) with the following conditions are satisfied:

$$u_4 < min\left\{\frac{1}{(u_1 + x^*)}, \frac{u_2\, x^{*2}}{(u_1 + x^*)(2\,y^* - 1)\ y^{*2}}\right\} \qquad (4.4\text{ a})$$

$$\frac{4B_1B_2 - B_1^3}{4} < B_3 < \frac{B_1B_2}{2} \qquad (4.4\text{ b})$$

Then at the parameter value $u_7 = u_7^*$, the system (2) has a Hopf bifurcation near the point E_3 .

Proof: Consider the characteristic equation of system (2) at E_3 which is given by eq. (10 b), then by using the Hopf bifurcation theorem, for n=4, we need to find a parameter say (u_7^*) to verify the necessary and sufficient conditions for the Hopf bifurcation to satisfy that: $B_i(u_7^*) > 0\ ;\ i = 1,3,4$, $\Delta_1(u_7^*) > 0$, $B_1^3(u_7^*) - 4\,\Delta_1(u_7^*) > 0$ and $\Delta_2(u_7^*) = 0$,where $B_I\ ;\ i = 1,3,4$ represents the coefficients of characteristic equation eq. (10 b).

Straight forward computation gives that:
$B_i(u_7^*) > 0$; $i = 1,3,4$ and $\Delta_1(u_7^*) > 0$ under the following locally conditions (10 c), (10 d) and (10 e) that given in [6] which are:

$y^* > \frac{1}{2}$,

$$w^* > \frac{u_1 u_4\, u_6 (1-m)^2 (2y^*-1) y^{*2}}{u_9 [u_1 (1-m) y^* z^* + [(u_2+u_3) y^* + u_2 x^*](u_1+x^*)^2]},$$

$$\frac{u_1 u_6 (2y^*-1)}{u_1(1-m)\, z^* + (u_2+u_3)(u_1+x^*)^2} < u_7 \quad < \frac{u_1 u_6 [u_1(1-m)\, z^* + (u_2+u_3)(u_1+x^*)^2]}{u_2\,(u_1+x^*)^4}$$

while $B_1^3\,(u_7^*) - 4\,\Delta_1(u_7^*) > 0$ provided that the condition (4.4 b) holds.

On the other hand, it is observed that $\Delta_2 = 0$ gives that:

$$B_3\,(B_1 B_2 - B_3) - B_1^2\, B_4 = 0$$

Straight forward computation we get:

$$N_1\, u_7^{*2} + N_2\, u_7^* + N_3 = 0\,, \qquad (4.4\text{ c})$$

where: $N_1 = P_1\,(1-m)^2\, z^{*2}$, $N_2 = P_2\,(1-m)\, z^*$,
$N_3 = \propto_0 (\propto_2 - \propto_1)[\, d_{22} \propto_3 - \propto_6] + [\, d_{11} \propto_3 + \propto_6][(\propto_0 \propto_5 - \propto_6) + d_{22} \propto_3]$,
with, $P_1 = (d_{11} d_{23} - d_{13} d_{21})(d_{22} d_{23} + d_{13} d_{21})$,
$P_2 = [\propto_0 (\propto_2 - \propto_1) + d_{11} \propto_3](d_{11} d_{23} - d_{13} d_{21}) + d_{22} d_{23} [(\propto_0 \propto_5 - \propto_6) + d_{22} \propto_3]$
$+ \propto_6 (d_{11} d_{23} - 2 d_{13} d_{21}) + d_{13} d_{21} (\propto_0 \propto_5 + d_{22} \propto_3)$.

Clearly, $N_1 < 0$ *and* $N_3 > 0$ *prvided that in addition to the locally conditions* (10 *c*) and (10 *d*) , the conditon (4.4 a) holds .

Note that, the conditon (4.4 a) guarantees that the last term of P_1 is negative while the first term of N_3 is positive.

So, the eq. (4.4 c) has a unique positive root:

$$u_7^* = \frac{1}{2N_1}\left(-N_2 + \sqrt{N_2^2 - 4N_1 N_3}\right)$$

Now, at $u_7 = u_7^*$ the characteristic equation given by eq. (10 *b*) given in [6] can be written as:

$\left(\lambda^2 + \frac{B_3}{B_1}\right)\left(\lambda^2 + B_1 \lambda + \frac{\Delta_1}{B_1}\right) = 0$, which has four roots,

$\lambda_{1,2} = \pm i \sqrt{\frac{B_3}{B_1}}$ and $\lambda_{3,4} = \frac{1}{2}\left(-B_1 \pm \sqrt{B_1^2 - 4\frac{\Delta_1}{B_1}}\right)$.

Clearly, at $u_7 = u_7^*$ there are two pure imaginary eigenvalues (λ_1 and λ_2) and two eigenvalues which are real and negative. Now for all values of u_7 in the neighborhood of u_7^* , the roots in general of the following form:

$$\lambda_1 = \varepsilon_1 + i\varepsilon_2\ , \lambda_2 = \varepsilon_1 - i\varepsilon_2\ ,\ \lambda_{3,4} = \frac{1}{2}\left(-B_1 \pm \sqrt{B_1^2 - 4\frac{\Delta_1}{B_1}}\right).$$

Clearly, $Re\big(\lambda_k(a_2)\big)\big|_{u_7 = u_7^*} = \varepsilon_1(u_7^*) = 0$, $k = 1,2$ that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $u_7 = u_7^*$.

Now, according to verify the transversality condition we must prove that:

$$\Theta^*(u_7^*)\, \Psi^*(u_7^*) + \Gamma^*(u_7^*)\, \Phi^*(u_7^*) \neq 0 \quad ,$$

where Θ^*, Ψ^*, Γ^* and Φ^* are given in (4.3). Note that for $u_7 = u_7^*$ we have $\varepsilon_1(u_7^*) = 0$ and $\varepsilon_2(u_7^*) = \sqrt{\frac{B_3}{B_1}}$, substituting into (4.3) gives the following simplifications:

$\Psi^*(u_7^*) = -2\, B_3(u_7^*)$, $\Phi^*(u_7^*) = 2\, \frac{\varepsilon_2(u_7^*)}{B_1}(B_1 B_2 - 2\, B_3)$,

$$\Theta^*(u_7^*) = B_4'(u_7^*) - \frac{B_3}{B_1} B_2'(u_7^*)\ ,$$

$$\Gamma^*(u_7^*) = \varepsilon_2(u_7^*)\left(B_3'(u_7^*) - \frac{B_3}{B_1} B_1'(u_7^*)\right),$$

where: $B_1' = \frac{dB_1}{du_7}\,|_{u_7=u_7^*} = 0$, $B_2' = \frac{dB_2}{du_7}\,|_{u_7=u_7^*} = -d_{23}\,(1-m)\,z^*$,
$B_3' = \frac{dB_3}{du_7}\,|_{u_7=u_7^*} = (d_{11}d_{23} - d_{13}d_{21})(1-m)\,z^*$, $B_4' = \frac{dB_4}{du_7}\,|_{u_7=u_7^*} = 0$.

Then by using eq. (4.4) we get that:

$$\Theta^*(u_7^*)\,\Psi^*(u_7^*) + \Gamma^*(u_7^*)\,\Phi^*(u_7^*) = P_3 + P_4 \neq 0,$$

where: $P_3 = \frac{-2d_{23}(1-m)\,z^*\,{B_3}^2}{B_1}$,

$$P_4 = \frac{2\,{\varepsilon_2}^2\,(u_7^*)(1-m)\,z^*(d_{11}d_{23} - d_{13}d_{21})(B_1B_2 - 2\,B_3)}{B_1}.$$

Now, according to condition (4.4 b) we have:

$$\Theta^*(u_7^*)\,\Psi^*(u_7^*) + \Gamma^*(u_7^*)\,\Phi^*(u_7^*) \neq 0\ .$$

So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_3 at the parameter $u_7 = u_7^*$.

***Numerical analysis of system*(2) [6]**

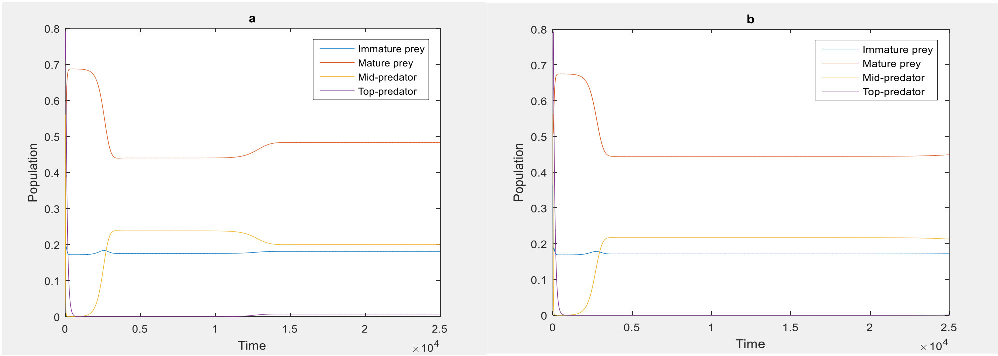
In this section, the dynamical behavior of system (2) is studied numerically for a set of parameters which is given by (5.1) and different sets of initial points which is given in [6]. Our obtained results were confirmed in the previous sections numerically by using Runge Kutta method along with predictor corrector method which represents the first objective of this numerical simulations study, while the second objective is to check the existence of the bifurcation near the equilibrium points which is given[6].

$$\left.\begin{array}{c}\boldsymbol{u_1 = 0.6\ , u_2 = 0.4\ , u_3 = 0.1\ , u_4 = 0.5\ , u_5 = 0.1\ , u_6 = 0.3\ ,}\\ \boldsymbol{u_7 = 0.3\ , u_8 = 0.1\ , u_9 = 0.5\ , u_{10} = 0.1\ ,\ m = 0.5\ .}\end{array}\right] \qquad (5.1)$$

System (2) is solved numerically for the data given in (5.1) with varying one parameter at each time which results the following outputs that represent the numerical bifurcation of system (2):

- By varying one of the parameters u_i , $i = 1,2,4$ and 6 (which represent the half saturation rate of mid-predator upon immature prey, the growth rate parameter of immature prey, the predation rate of mid-predator upon immature prey and the conversion rate from immature prey to mid-predator respectively) each time and keeping the rest of parameters as data given in (5.1) results that the solution of system (2) approaches asymptotically to the positive equilibrium point E_3, on the other word these parameters did not play a vital role in the bifurcation analysis of system (2) within the set of parameters given in (5.1). (For more details see [6]).

Varying the natural death rate of immature prey parameter u_3 in the range $0.01 \leq u_3 < 0.90$ it is observed that the solution of system (2.2) approaches asymptotically to the positive equilibrium point E_3 , while increasing this parameter for $0.90 \leq u_3 < 1$ causes that the solution of system (2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of xyz − space, thus, the parameter u_3 when $u_3 = 0.90$ is a bifurcation point as shown in Fig.(5.1) for the typical value of $u_3 = 0.85$ and the bifurcation point $u_3 = 0.90$.



Fig(5.1): - (a): - Time series of the solution of system (2) for the data given by (5.1) with $u_3 = 0.85$ which approaches to $E_3 = (0.18, 0.48, 0.2, 0.007)$ in the interior of R_+^4 , **(b):** Time series of the solution of system (2) for the data given by

(5.1)with the bifurcation point $u_3 = 0.90$ which approaches to $E_2 = (0.17, 0.44, 0.2, 0)$ in the interior of the positive quadrant of $xyz-$space .

- Varying the mature prey natural death rate parameter u_5 in the range $0.01 \le u_5 < 0.41$ causes that the solution of system (2) approaches asymptotically to a positive equilibrium point E_3, however increasing this parameter in the range $0.41 \le u_5 < 0.58$ causes extinction in the top-predator and the solution of system (2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz-$space, further increasing in the range $0.58 \le u_5 < 0.8$ causes extinction in the mid-predator and the solution of system (2) approaches asymptotically to the free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ in the interior of the positive quadrant of $xy-$plane , then more increasing of this parameter in the range $0.8 \le u_5 < 1$ causes extinction in all species and the solution of system (2) approaches asymptotically to the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$, thus, the parameter u_5 when $u_5 = 0.41$, $u_5 = 0.58$ and $u_5 = 0.8$ is a bifurcation point as shown in Fig.(5.2) for the typical value of $u_5 = 0.38$ and the bifurcation point $u_5 = 0.41$.

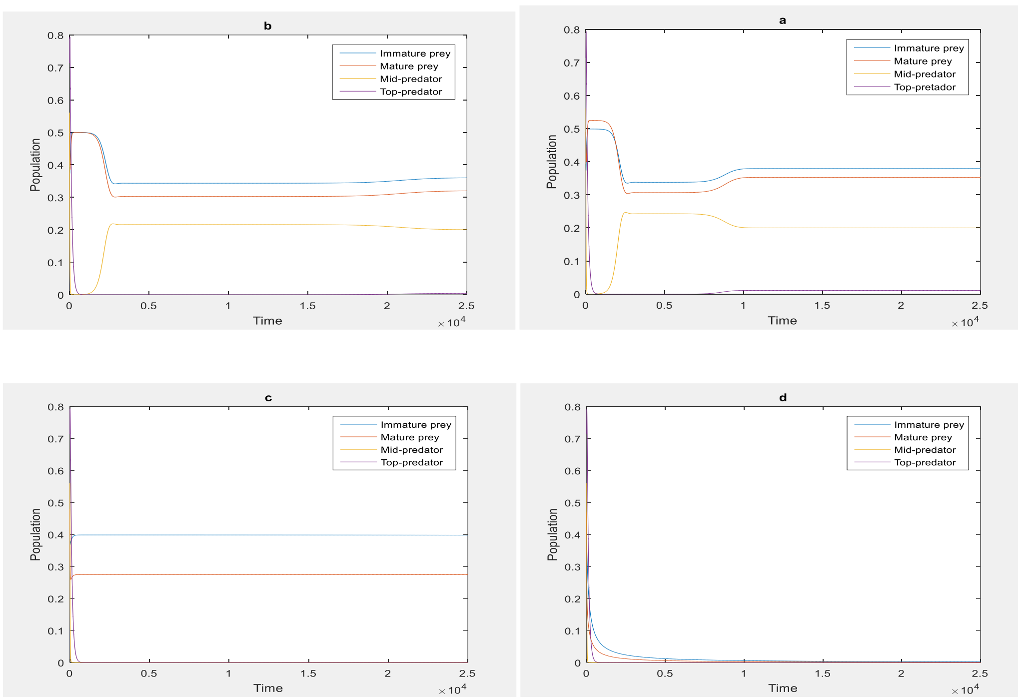
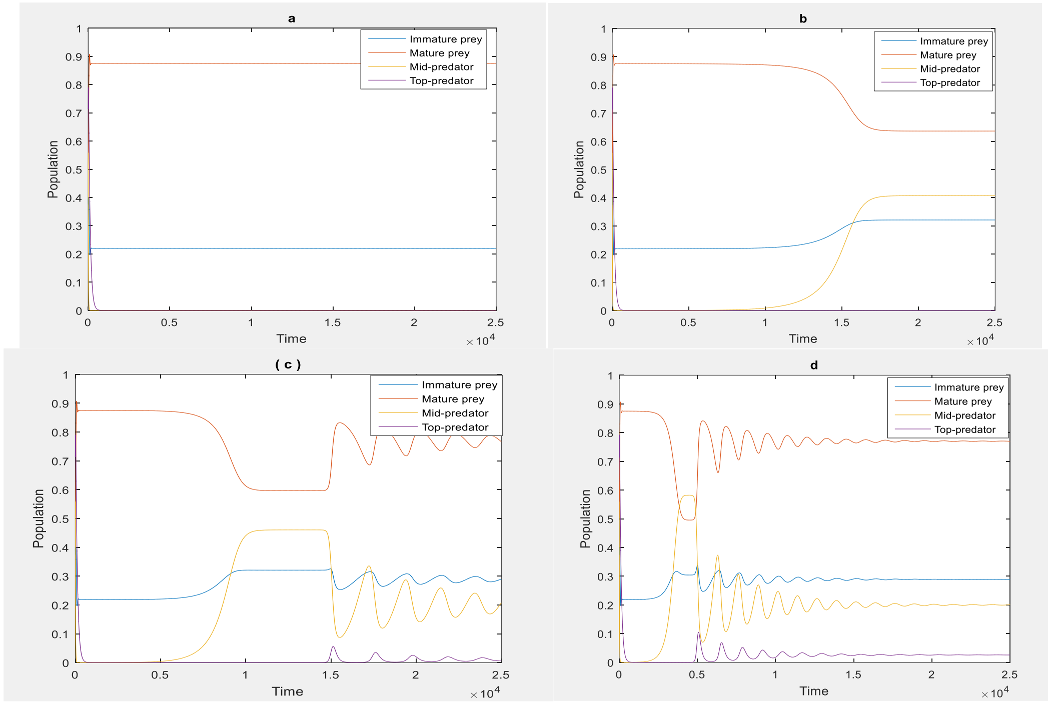


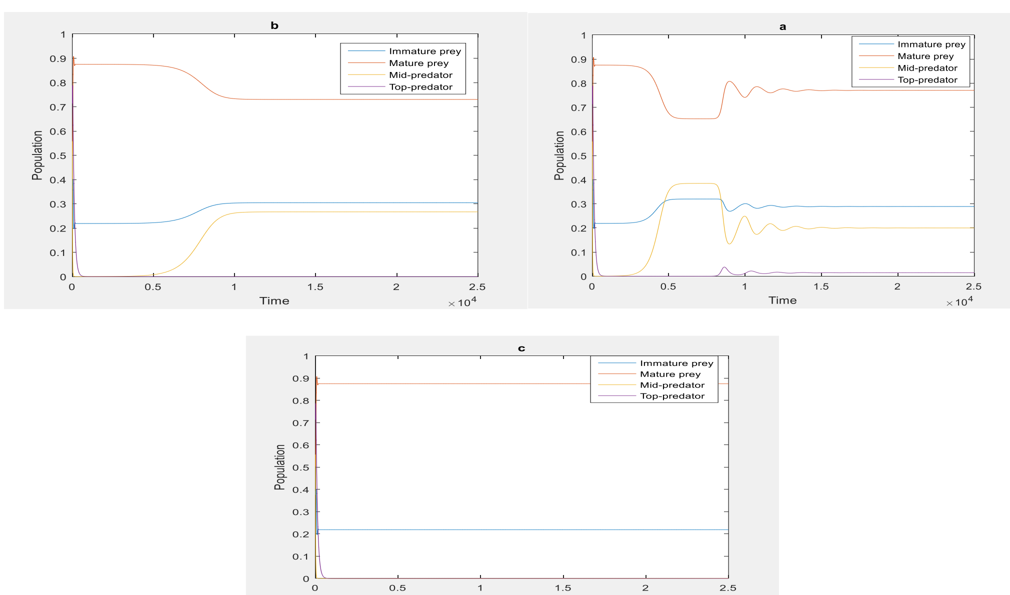
Fig (5.2): - (a): Time series of the solution of system (2) for the data given by (5.1) with $u_5 = 0.38$, approaches to $E_3 = (0.37, 0.35, 0.2, 0.01)$ in the interior of R^4_+, (**b**): Time series of the solution of system (2) for data given by (5.1) with the bifurcation point $u_5 = 0.41$ approaches asymptotically stable to $E_2 = (0.36, 0.31, 0.2, 0)$ in the positive quadrant of $xyz-$space, (**c**): Time series of the solution of system (2) for data given by (5.1) with the bifurcation point $u_5 = 0.58$ approaches asymptotically stable to $E_2 = (0.39, 0.27, 0, 0)$, (**d**):Time series of the solution of system (2) for data given by (5.1) with the bifurcation point $u_5 = 0.8$ approaches asymptotically stable to $E_0 = (0, 0, 0, 0)$.

- For varying the conversion rate parameter from the mature prey to the mid-predator u_7, with $0.01 \le u_7 < 0.15$ the solution of system (2) approaches asymptotically to the positive free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ in the interior of the positive quadrant of $xy-$plane, , while for $u_7 = 0.15$ the solution of system (2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz-space$, which means revival of the mid-predator population, then increasing this parameter in the range $0.15 \le u_7 < 0.20$ leads revival of the top-predator and a small periodic attractor appears, for more increasing in the range $0.20 \le u_7 < 0.5$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_3 , thus, the parameter u_7 when $u_7 = 0.15$ and $u_7 = 0.20$ is a bifurcation point as shown in Fig.(5.3) for the bifurcation points $u_7 = 0.15$, $u_7 = 0.20$ and the typical values $u_7 = 0.16$ and $u_7 = 0.19$.



Fig(5. 3):- Time series of the solution of system (2) for the data given by (5.1) with different values of u_7, **(a):** $E_1 = (0.21, 0.87, 0, 0)$ is a asymptotically stable with $u_7 = 0.14$, **(b):** $E_2 = (0.32, 0.63, 0.4, 0)$ is a asymptotically stable with $u_7 = 0.15$, **(c):** periodic attractor with $u_7 = 0.16$, **(d):** $E_3 = (0.28, 0.77, 0.2, 0.02)$ is a asymptotically stable with the bifurcation point $u_7 = 0.20$.

- The varying of the mid-predator natural death rate parameter u_8 in the range $0.01 \le u_8 < 0.16$ the solution of system(2) approaches asymptotically to the positive equilibrium point E_3, further increasing of this parameter with $u_8 = 0.16$ which causes extinction in the top-predator and the solution of system (2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz -$ space, while for $0.17 \le u_8 < 1$ causes the extinction of the mid –predator the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$ in the interior of the positive quadrant of $xy -$ plane, thus, the parameter u_8 when $u_8 = 0.16$ and $u_8 = 0.17$ is a bifurcation point as shown in Fig.(5.4) , for typical value $u_8 = 0.15$ and the bifurcation points $u_8 = 0.16$ and $u_8 = 0.17$.



Fig(5.4) (a): - Time series of the solution of system (2) for the data given by (5.1) with $u_8 = 0.15$, which approaches to $E_3 = (0.28, 0.77, 0.2, 0.01)$, (b): Time series of the solution of system (2) for the data given by (5.1) with $u_8 = 0.16$, which approaches to $E_2 = (0.30, 0.73, 0.26, 0)$ in the interior of the positive quadrant of $xyz - space$, (c): Time series of the solution of system (2) for the data given by (5.1) with $u_8 = 0.17$, which approaches to $E_1 = (0.21, 0.87, 0, 0)$ in the interior of the positive quadrant of $xy - plane$.

- Varying of the conversion rate of predation parameter of the top-predator upon the mid-predator in the range $0.01 \leq u_9 < 0.15$ the solution of system (2.2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz - space$, while for $0.15 \leq u_9 < 1$, the top-predator population revives and the solution of system (2) approaches asymptotically to a positive equilibrium point E_3, as shown in as shown in Figs.(6.7) a and (6.7) b , for typical values 0f u_9 (see [6]).
- The increasing the natural death rate of top-predator parameter u_{10} in the range $0.1 \leq u_{10} < 0.35$, the solution of system (2) approaches asymptotically to a positive equilibrium point E_3, while the increasing of this parameter for $0.35 \leq u_{10} < 1$ causes extinction of the top-predator population and the solution of system (2) approaches asymptotically to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz - space$, thus, the parameter u_{10} when $u_{10} = 0.35$ is a bifurcation point as shown in Fig.(5.5) , for typical value $u_{10} = 0.33$ and the bifurcation point $u_{10} = 0.35$.

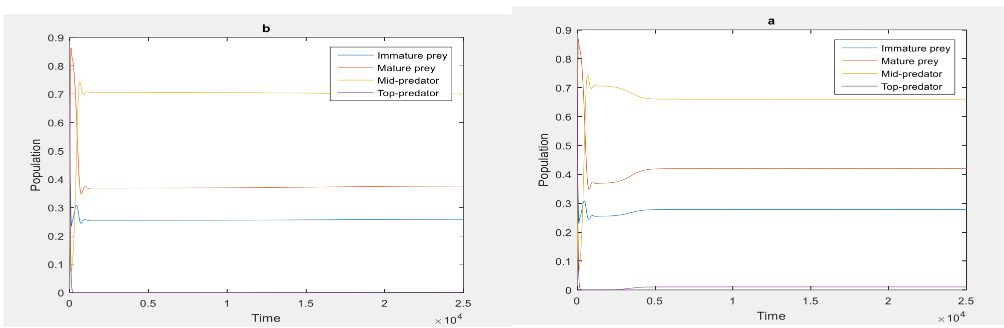


Fig.(5.5):- (a): Time series of the solution of system (2) for the data given by (5.1) with $u_{10} = 0.33$, which approaches to $E_3 = (0.27, 0.41, 0.66, 0.01)$in the interior of R^4_+ . **(b):** Time series of the solution of system (2) for the data given by (5.1) with $u_{10} = 0.35$, which approaches to $E_2 = (0.25, 0.37, 0.7, 0)$ in the interior of the positive quadrant of $xyz - space$.

- Finally, varying the number of prey inside the refuge parameter m and keeping the rest of parameters values as data given in (5.1), it is observed that for $0.01 \leq m < 0.63$ the solution of system (2) approaches asymptotically to the positive equilibrium point E_3, while increasing this parameter in the range $0.63 \leq m < 0.69$ leads that the solution of system (2) approaches asymptotically to a periodic dynamics in Int. R^4_+, more increasing of this parameter in the range $0.69 \leq m < 0.71$ causes extinction of the top-predator population and the solution of system (2) approaches asymptotically to the free top-predator equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ in the interior of the positive quadrant of $xyz - space$, for $0.71 \leq m < 1$ the solution of system (2) approaches asymptotically to the free predators' equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ in the interior of the positive quadrant of $xy - plane$, thus, the parameter m when $m = 0.63$, $m = 0.69$ and $m = 0.71$ is a bifurcation point as shown in Fig.(5.6), for typical value $m = 0.68$ and the bifurcation point $m = 0.69$. (more details see [6]).

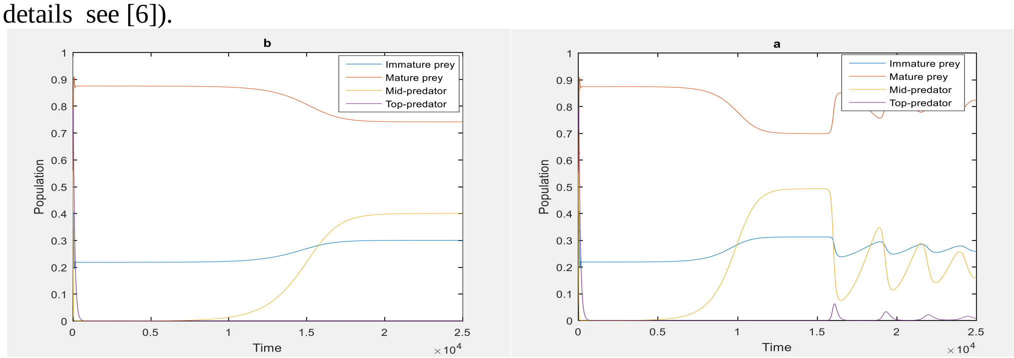


Fig.(5.6):- (a): Time series of the solution of system (2) for the data given by (5.1) with $m = 0.68$, which approaches asymptotically to a periodic dynamics in the interior of R^4_+ . (b): Time series of the solution of system (2)for the data given by (5.1) with $m = 0.69$, which approaches to $E_2 = (0.30, 0.74, 0.4, 0)$ in the interior of the positive quadrant of $xyz - space$.

DISCUSSION & CONCLUSION

In this paper, an ecological model that described the dynamical behavior of the food chain real system has been proposed and analyzed. The model included four nonlinear autonomous differential equations that describe the dynamics of four different population, namely first immature prey (x), mature prey (y), mid-predator (z) and (w) which is represent the top-predator. By the application of the Sotomayor's theorem the necessary conditions for the local bifurcation around each equilibrium point and a Hopf bifurcation near the positive equilibrium point E_3 were established analytically which have been demonstrated the occurrence of:

- A transcritical bifurcation around E_0 .
- Either a transcritical or a pitchfork bifurcation around E_1 and E_2 .
- A saddle-node and a Hopf bifurcation round E_3.

Finally, numerical simulation has been used to specific the control set of parameters that affect dynamics of the system and confirm our obtained analytical results. Therefore system (2) has been solved numerically for different sets of initial points and a set of parameters starting with the hypothetical set of data given by eq. (5.1) and the following observations are obtained.

1- System (2) has two types of attractor in Int. R_+^4 either a stable point or a periodic attractor.

2- For the set hypothetical parameters value given in eq. (5.1), it is observed that varying the parameter values; $u_{i,}i = 1,2,4 \text{ and } 6$ do not have any effect on the dynamical behavior of system (2) and the solution of the system (2) still approaches to positive equilibrium point $E_3 = (x^*, y^*, z^*, w^*)$.

3- As the natural death rate of immature prey u_3 increasing to 0.89 keeping the rest of parameters as in eq. (5.1), the solution of system (2) approaches to positive equilibrium point E_3. However if $0.90 \le u_3 < 1$, then the top-predator will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, thus, the parameter $u_3 = 0.9$ is a bifurcation point.

4- As the natural death rate of mature prey u_5 increasing to 0.40 keeping the rest of parameters as in eq.(5.1), the solution of system (2) approaches to positive equilibrium E_3, however if $0.41 \le u_5 < 0.58$, then the top-predator will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, further increasing in the range $0.58 \le u_5 < 0.8$ causes the mid-predator faced extinction in and then the trajectory transferred from the free top-predator's equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, to the free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, then more increasing of this parameter in the range $0.8 \le u_5 < 1$ causes extinction in all species and then the trajectory transferred from equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$,to the vanishing equilibrium point $E_0 = (0, 0, 0, 0)$,thus, the parameter u_5 when $u_5 = 0.41, u_5 = 0.58$ and $u_5 = 0.8$ is a bifurcation point.

5- As the parameter u_7 which represents the conversion rate from the mature prey to the mid-predator decreasing to 0.15 keeping the rest of parameters as in eq. (5.1), the solution of system (2) approaches to the positive free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, while for the $u_7 = 0.15$, then the trajectory transferred from the free predators equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$ to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, which means revival of the mid-predator population, then increasing this parameter in the range $0.15 \le u_7 < 0.20$ leads revival of the top-predator and the trajectory approaches asymptotically to a periodic dynamics in Int. R_+^4 , for more increasing in the range $0.20 \le u_7 < 0.5$, the trajectory will transferred asymptotically from a periodic dynamics in Int. R_+^4 and then approaches asymptotically stable to a positive equilibrium point $E_3(x^*, y^*, z^*, w^*)$, thus, the parameter u_7 when $u_7 = 0.15$ and $u_7 = 0.20$ is a bifurcation point.

6- As the natural death rate of the mid-predator u_8 increasing to 0.15 keeping the rest of parameters as in eq. (5.1), the solution of system (2) approaches to the positive equilibrium point E_3, further increasing in the range $0.16 \le u_8 < 0.18$ causes the top-predator faced extinction and the trajectory transferred from the positive equilibrium point E_3 to the free top-predator equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, but for $0.18 \le u_8 < 1$ causes the mid – predator faced extinction and the trajectory transferred from $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ to $E_1 = (\bar{x}, \bar{y}, 0, 0)$, thus, the parameter u_8 when $u_8 = 0.16$ and $u_8 = 0.18$ is a bifurcation point.

7- As the conversion rate of predation parameter of the top-predator upon the mid-predator u_9 increasing to 0.14 keeping the rest of parameters as in eq. (5.1), the solution of system (2) approaches the free top-predator equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, while for $0.15 \le u_9 < 1$ the top-predator population revives and then the trajectory transferred from the point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$ to the positive equilibrium point E_3, thus, the parameter $u_9 = 0.15$ is a bifurcation point.

8- As the natural death rate of the top-predator parameter u_{10}increasing in the range $0.1 \le u_{10} < 0.35$ keeping the rest of parameters as in eq. (5.1), the solution of system (2) approaches asymptotically to a positive equilibrium point E_3, while increasing this parameter in the range $0.35 \le u_{10} < 1$ causes extinction of the top-predator population and then the trajectory transferred from positive equilibrium point to $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, thus, the parameter $u_{10} = 0.35$ is a bifurcation point.

9- As the number of prey inside the refuge parameter m varying in the range $0.01 \le m < 0.63$ and keeping the rest of parameters values as data given in eq. (5.1), the solution of system (2) approaches asymptotically to the positive equilibrium point E_3, while increasing this parameter in the range $0.63 \le m < 0.69$ leads that the trajectory approaches asymptotically to a periodic dynamics in Int. R_+^4, while increasing this parameter for $0.69 \le m < 0.71$ causes extinction of the top-predator population and restore the stability and then the trajectory transferred asymptotically from a periodic dynamics in Int. R_+^4 to the stable free top-predator equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, then more increasing of this parameter for $0.71 \leq m$ causes extinction of the mid-predator population and the trajectory transferred from $E_2 = (\hat{x}, \hat{y}, \hat{z}, 0)$, to $E_1 = (\bar{x}, \bar{y}, 0, 0)$, thus, the parameter m when m = 0.63 , m = 0.69 and m = 0.71 is a bifurcation point.

REFERENCES

[1]. Kuznetsov, Y.A. (2006): Andronov-Hopf bifurcation, Scholarpedia, 1(10):1858.

[2]. Crawford, J.D., Kueny, C., Saphir, B., Shadwick, B., (1989): Introduction to bifurcation theory, Texas Univ., Austin, TX (USA), Inst. for Fusion Studies.

[3]. Naji, R.K. and Majeed, S.J. (2016) A prey-predator model with a refuge-stage structure prey population, International Journal of Differential Equations, Volume 2016 Article ID 2010464, 10 pages.

[4]. Kadhim, Z. J., Majeed, A.A. and Naji, R. K. (2016) The bifurcation analysis of a stage structured prey food web model with refuge, Iraqi Journal of Science, Special Issue, Part A, pp:139-155.

[5]. Majeed, A. A. and Ali, N.H. (2016) The bifurcation of stage structured prey-predator food chain model with refuge, International Journal of Modern Research in Engineering and Technology (IJMRET), Volume 1 Issue 5.

[6]. Majeed, A.A. and Rahi, S.A. (2017) A food chain structures and dynamics across a refuge stage- structure prey-predator model, International Journal of Advanced Research (IJAR), Ady. Res. 5(3), 1362-1380.

[7]. *Lawrence*, P. (2001) Differential equation and dynamical system, third edition, New York, Springer-Verlag Inc.

[8]. Haque, M. and Venturino, E. (2006): Increase of the prey may decrease the healthy predator population in presence of disease in the predator, HERMIS, 7, 38-59.